Part 3 General Relativity

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Contents

Preface

These are lecture notes for the course on General Relativity in Part III of the Cambridge Mathematical Tripos. There are introductory GR courses in Part II (Mathematics or Natural Sciences) so, although self-contained, this course does not cover topics usually covered in a first course, e.g., the Schwarzschild solution, the solar system tests, and cosmological solutions. You should consult an introductory book (e.g. *Gravity* by J.B. Hartle) if you have not studied these topics before.

Acknowledgment

I am very grateful to Andrius Štikonas for producing the figures.

Conventions

Apart from the first lecture, we will use units in which the speed of light is one: $c = 1$.

We will use "abstract indices" a, b, c etc to denote tensors, e.g. V^a, g_{cd}. Equations involving such indices are basis-independent. Greek indices μ, ν etc refer to tensor components in a particular basis. Equations involving such indices are valid only in that basis.

We will define the metric tensor to have signature $(-+++)$, which is the most common convention. Some authors use signature $(+---)$.

Our convention for the Riemann tensor is such that the Ricci identity takes the form

$$\nabla_a \nabla_b V^c - \nabla_b \nabla_a V^c = R^c{}_{dab} V^d.$$

Some authors define the Riemann tensor with the opposite sign.

Bibliography

There are many excellent books on General Relativity. The following is an incomplete list:

1. *General Relativity*, R.M. Wald, Chicago UP, 1984.

2. *Advanced General Relativity*, J.M. Stewart, CUP, 1993.

3. *Spacetime and geometry: an introduction to General Relativity*, S.M. Carroll, Addison-Wesley, 2004.

4. *Gravitation*, C.W. Misner, K.S. Thorne and J.A. Wheeler, Freeman 1973.

5. *Gravitation and Cosmology*, S. Weinberg, Wiley, 1972.

Our approach will be closest to that of Wald. The first part of Stewart's book is based on a previous version of this course. Carroll's book is a very readable introduction. Weinberg's book contains a good discussion of equivalence principles. Our treatments of the Newtonian approximation and gravitational radiation are based on Misner, Thorne and Wheeler.

Chapter 1

Equivalence Principles

1.1 Incompatibility of Newtonian gravity and Special Relativity

Special relativity has a preferred class of observers: inertial (non-accelerating) observers. Associated to any such observer is a set of coordinates (t, x, y, z) called an *inertial frame*. Different inertial frames are related by Lorentz transformations. The Principle of Relativity states that physical laws should take the same form in any inertial frame.

Newton's law of gravitation is

$$\nabla^2 \Phi = 4\pi G \rho \tag{1.1}$$

where Φ is the gravitational potential and ρ the mass density. Lorentz transformations mix up time and space coordinates. Hence if we transform to another inertial frame then the resulting equation would involve time derivatives. Therefore the above equation does not take the same form in every inertial frame. Newtonian gravity is incompatible with special relativity.

Another way of seeing this is to look at the solution of (1.1):

$$\Phi(t, \mathbf{x}) = -G \int d^3\mathbf{y} \frac{\rho(t, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \tag{1.2}$$

From this we see that the value of Φ at point $\mathbf{x}$ will respond *instantaneously* to a change in ρ at point $\mathbf{y}$. This violates the relativity principle: events which are simultaneous (and spatially separated) in one inertial frame won't be simultaneous in all other inertial frames.

The incompatibility of Newtonian gravity with the relativity principle is not a problem provided all objects are moving non-relativistically (i.e. with speeds much less than the speed of light c). Under such circumstances, e.g. in the Solar System, Newtonian theory is very accurate.

Newtonian theory also breaks down when the gravitational field becomes strong. Consider a particle moving in a circular orbit of radius r about a spherical body of mass M, so $\Phi = -GM/r$. Newton's second law gives $v^2/r = GM/r^2$ hence $v^2/c^2 = |\Phi|/c^2$. Newtonian theory requires non-relativistic motion, which is the case only if the gravitational field is weak: $|\Phi|/c^2 \ll 1$. In the Solar System $|\Phi|/c^2 < 10^{-5}$.

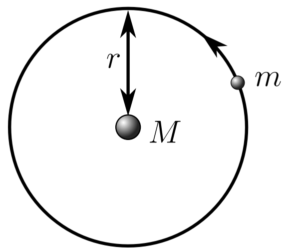


Figure 1.1: Circular orbit

GR is the theory that replaces both Newtonian gravity and special relativity.

1.2 The weak equivalence principle

The equivalence principle was an important step in the development of GR. There are several forms of the EP, which are motivated by thought experiments involving Newtonian gravity. (If we consider only experiments in which all objects move non-relativistically then the incompatibility of Newtonian gravity with the relativity principle is not a problem.)

In Newtonian theory, one can distinguish between the notions of *inertial* mass m_I, which appears in Newton's second law: $\mathbf{F} = m_I \mathbf{a}$, and *gravitational* mass, which governs how a body interacts with a gravitational field: $\mathbf{F} = m_G \mathbf{g}$. Note that this equation defines both m_G and $\mathbf{g}$ hence there is a scaling ambiguity $\mathbf{g} \to \lambda \mathbf{g}$ and $m_G \to \lambda^{-1} m_G$ (for all bodies). We fix this by defining $m_I/m_G = 1$ for a particular test mass, e.g., one made of platinum. Experimentally it is found that other bodies made of other materials have $m_I/m_G - 1 = \mathcal{O}(10^{-12})$ (Eötvös experiment).

The exact equality of m_I and m_G for all bodies is one form of the *weak equivalence principle*. Newtonian theory provides no explanation of this equality.

The Newtonian equation of motion of a body in a gravitational field $\mathbf{g}(\mathbf{x}, t)$ is

$$m_I \ddot{\mathbf{x}} = m_G \mathbf{g}(\mathbf{x}(t), t) \tag{1.3}$$

using the weak EP, this reduces to

$$\ddot{\mathbf{x}} = \mathbf{g}(\mathbf{x}(t), t) \tag{1.4}$$

Solutions of this equation are uniquely determined by the initial position and velocity of the particle. Any two particles with the same initial position and velocity will follow the same trajectory. This means that the weak EP can be restated as: *The trajectory of a freely falling test body depends only on its initial position and velocity, and is independent of its composition.*

By "test body" we mean an uncharged object whose gravitational self-interaction is negligible, and whose size is much less than the length over which external fields such as $\mathbf{g}$ vary.

Consider a new frame of reference moving with constant acceleration $\mathbf{a}$ with respect to the first frame. The origin of the new frame has position $\mathbf{X}(t)$ where $\ddot{\mathbf{X}} = \mathbf{a}$. The coordinates of the new frame are $t' = t$ and $\mathbf{x}' = \mathbf{x} - \mathbf{X}(t)$. Hence the equation of motion in this frame is

$$\ddot{\mathbf{x}}' = \mathbf{g} - \mathbf{a} \equiv \mathbf{g}' \tag{1.5}$$

The motion in the accelerating frame is the same as in the first frame but with a different gravitational field $\mathbf{g}'$. If $\mathbf{g} = 0$ then the new frame appears to contain a uniform gravitational field $\mathbf{g}' = -\mathbf{a}$: *uniform acceleration is indistinguishable from a uniform gravitational field.*

Consider the case in which $\mathbf{g}$ is constant and non-zero. We can define an inertial frame as a reference frame in which the laws of physics take the simplest form. In the present case, it is clear that this is a frame with $\mathbf{a} = \mathbf{g}$, i.e., a freely falling frame. This gives $\mathbf{g}' = 0$ so an observer at rest in such a frame, i.e., a freely falling observer, does not observe any gravitational field. From the perspective of such an observer, the gravitational field present in the original frame arises because this latter frame is accelerating with acceleration $-\mathbf{g}$ relative to him.

Even if the gravitational field is not uniform, it can be approximated as uniform for experiments performed in a region of space-time sufficiently small that the non-uniformity is negligible. In the presence of a non-constant gravitational field, we define a *local inertial frame* to be a set of coordinates (t, x, y, z) that a freely falling observer would define in the same way as coordinates are defined in Minkowski spacetime. The word *local* emphasizes the restriction to a small region of spacetime, i.e., t, x, y, z are restricted to sufficiently small values that any variation in the gravitational field is negligible.

In a local inertial frame, the motion of test bodies is indistinguishable from the motion of test bodies in an inertial frame in Minkowski spacetime.

1.3 The Einstein equivalence principle

The weak EP governs the motion of test bodies but it does not tell us anything about, say, hydrodynamics, or charged particles interacting with an electromagnetic field. Einstein extended the weak EP as follows:
(i) The weak EP is valid. (ii) In a local inertial frame, the results of all non-gravitational experiments will be indistinguishable from the results of the same experiments performed in an inertial frame in Minkowski spacetime.

The weak EP implies that (ii) is valid for test bodies. But any realistic test body is made from ordinary matter, composed of electrons and nuclei interacting via the electromagnetic force. Nuclei are composed of protons and neutrons, which are in turn composed of quarks and gluons, interacting via the strong nuclear force. A significant fraction of the nuclear mass arises from binding energy. Thus the fact that the motion of test bodies is consistent with (ii) is evidence that the electromagnetic and nuclear forces also obey (ii). In fact *Schiff's conjecture* states that the weak EP implies the Einstein EP.

Note that we have motivated the Einstein EP by Newtonian arguments. Since we restricted to velocities much less than the speed of light, the incompatibility of Newtonian theory with special relativity is not a problem. But the Einstein EP is supposed to be more general than Newtonian theory. It is a guiding principle for the construction of a relativistic theory of gravity. In particular, any theory satisfying the EP should have some notion of "local inertial frame".

1.4 Tidal forces

The word "local" is essential in the above statement of the Einstein EP.

Consider a lab, freely falling radially towards the Earth, that contains two test particles at the same distance from the Earth but separated horizontally:

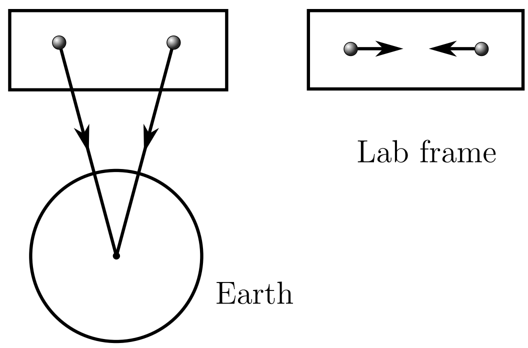


Figure 1.2: Tidal forces

The gravitational attraction of the particles is tiny and can be neglected. Nevertheless, as the lab falls towards Earth, the particles will accelerate towards each other because the gravitational field has a slightly different direction at the location of the two particles. This is an example of a *tidal force*: a force arising from non-uniformity of the gravitational field. Such forces are physical: they cannot be eliminated by free fall.

1.5 Bending of light

The Einstein EP implies that light is bent by a gravitational field.

Consider a uniform gravitational field again, e.g. a small region near the Earth's surface. A freely falling laboratory is a local inertial frame.

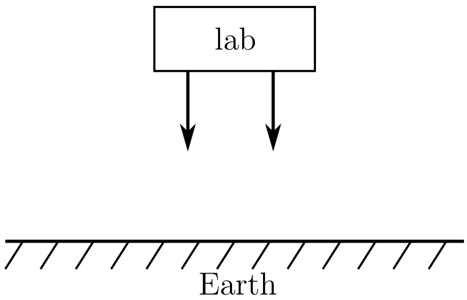


Figure 1.3: Freely falling lab near Earth's surface

Inside the lab, the Einstein EP tells us that light rays must move on straight lines. But a straight line with respect to the lab corresponds to a curved path w.r.t to a frame at rest relative to the Earth.

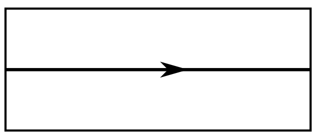


Figure 1.4: Light ray in the lab frame.

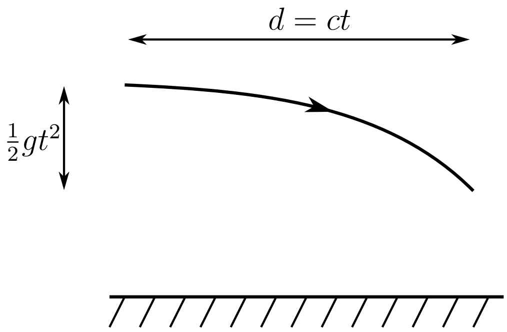


Figure 1.5: Light ray in Earth frame.

This shows that light falls in the gravitational field in exactly the same way as a massive test particle: in time t is falls a distance $(1/2)gt^2$. (The effect is tiny: if the field is vertical then the time taken for the light to travel a horizontal distance d is $t = d/c$. In this time, the light falls a distance $h = gd^2/(2c^2)$. Taking $d = 1\,\text{km}$, $g \approx 10\text{ms}^{-2}$ gives $h \approx 5 \times 10^{-11}\text{m}$.)

NB: this is a *local* effect in which the gravitational field is approximated as uniform so the result follows from the EP. It can't be used to calculate the bending of light rays by a non-uniform gravitational field e.g. light bending by the the Sun.

1.6 Gravitational red shift

Alice and Bob are at rest in a uniform gravitational field of strength g in the negative z-direction. Alice is at height $z = h$, Bob is at $z = 0$ (both are on the z-axis). They have identical clocks. Alice sends light signals to Bob at constant proper time intervals which she measures to be $\Delta\tau_A$. What is the proper time interval $\Delta\tau_B$ between the signals received by Bob?

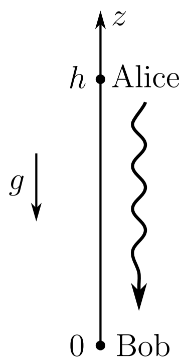


Figure 1.6: Pound-Rebka experiment

Alice and Bob both have acceleration g with respect to a freely falling frame. Hence, by the EP, this experiment should give identical results to one in which Alice and Bob are moving with acceleration g in the positive z-direction in Minkowski spacetime. We choose our freely falling frame so that Alice and Bob are at rest at $t = 0$.

We shall neglect special relativistic effects in this problem, i.e., effects of order v^2/c^2 where v is a typical velocity (the analysis can extended to include such effects). The trajectories of Alice and Bob are therefore the usual Newtonian ones:

$$z_A(t) = h + \frac{1}{2}gt^2, \qquad z_B(t) = \frac{1}{2}gt^2 \tag{1.6}$$

Alice and Bob have $v = gt$ so we shall assume that gt/c is small over the time it takes to perform the experiment. We shall neglect effects of order g^2t^2/c^2.

Assume Alice emits the first light signal at $t = t_1$. Its trajectory is $z = z_A(t_1) - c(t - t_1) = h + (1/2)gt_1^2 - c(t - t_1)$ so it reaches Bob at time $t = T_1$ where this equals $z_B(T_1)$, i.e.,

$$h + \frac{1}{2}gt_1^2 - c(T_1 - t_1) = \frac{1}{2}gT_1^2 \tag{1.7}$$

The second light signal is emitted at time $t = t_1 + \Delta\tau_A$ (there is no special relativistic time dilation to the accuracy we are using here so the proper time interval $\Delta\tau_A$ is the same as an inertial time interval). Its trajectory is $z = z_A(t_1 + \Delta\tau_A) - c(t - t_1 - \Delta\tau_A)$. Let it reach Bob at time $t = T_1 + \Delta\tau_B$, i.e., the proper time intervals between the signals received by Bob is $\Delta\tau_B$. Then we have

$$h + \frac{1}{2}g(t_1 + \Delta\tau_A)^2 - c(T_1 + \Delta\tau_B - t_1 - \Delta\tau_A) = \frac{1}{2}g(T_1 + \Delta\tau_B)^2. \tag{1.8}$$

Subtracting equation (1.7) gives

$$c(\Delta\tau_A - \Delta\tau_B) + \frac{g}{2}\Delta\tau_A(2t_1 + \Delta\tau_A) = \frac{g}{2}\Delta\tau_B(2T_1 + \Delta\tau_B) \tag{1.9}$$

The terms quadratic in $\Delta\tau_A$ and $\Delta\tau_B$ are negligible. This is because we must assume $g\Delta\tau_A \ll c$, since otherwise Alice would reach relativistic speeds by the time she emitted the second signal. Similarly for $\Delta\tau_B$.

We are now left with a linear equation relating $\Delta\tau_A$ and $\Delta\tau_B$

$$c(\Delta\tau_A - \Delta\tau_B) + g\Delta\tau_A t_1 = g\Delta\tau_B T_1 \tag{1.10}$$

Rearranging:

$$\Delta\tau_B = \left(1 + \frac{gT_1}{c}\right)^{-1}\left(1 + \frac{gt_1}{c}\right)\Delta\tau_A \approx \left(1 - \frac{g(T_1 - t_1)}{c}\right)\Delta\tau_A \tag{1.11}$$

where we have used the binomial expansion and neglected terms of order $g^2T_1^2/c^2$. Finally, to leading order we have $T_1 - t_1 = h/c$ (this is the time it takes the light to travel from A to B) and hence

$$\Delta\tau_B \approx \left(1 - \frac{gh}{c^2}\right)\Delta\tau_A \tag{1.12}$$

The proper time between the signals received by Bob is *less* than that between the signals emitted by Alice. Time appears to run more slowly for Bob. For example, Bob will see that Alice ages more rapidly than him.

If Alice sends a pulse of light to Bob then we can apply the above argument to each successive wavecrest, i.e., $\Delta\tau_A$ is the period of the light waves. Hence

$\Delta\tau_A = \lambda_A/c$ where λ_A is the wavelength of the light emitted by Alice. Bob receives light with wavelength λ_B where $\Delta\tau_B = \lambda_B/c$. Hence we have

$$\lambda_B \approx \left(1 - \frac{gh}{c^2}\right)\lambda_A. \tag{1.13}$$

The light received by Bob has shorter wavelength than the light emitted by Alice: it has undergone a *blueshift*. Light falling in a gravitational field is blueshifted.

This prediction of the EP was confirmed experimentally by the Pound-Rebka experiment (1960) in which light was emitted at the top of a tower and absorbed at the bottom. High accuracy was needed since $gh/c^2 = \mathcal{O}(10^{-15})$.

An identical argument reveals that light climbing out of a gravitational field undergoes a *redshift*. We can write the above formula in a form that applies to both situations:

$$\Delta\tau_B \approx \left(1 + \frac{\Phi_B - \Phi_A}{c^2}\right)\Delta\tau_A \tag{1.14}$$

where Φ is the gravitational potential. Our derivation of this result, using the EP, is valid only for *uniform* gravitational fields. However, we will see that GR predicts that this result is valid also for weak non-uniform fields.

1.7 Curved spacetime

The weak EP states that if two test bodies initially have the same position and velocity then they will follow exactly the same trajectory in a gravitational field, even if they have very different composition. (This is *not* true of other forces: in an electromagnetic field, bodies with different charge to mass ratio will follow different trajectories.) This suggested to Einstein that the trajectories of test bodies in a gravitational field are determined by the structure of spacetime alone and hence gravity should be described geometrically.

To see the idea, consider a spacetime in which the proper time between two infinitesimally nearby events is given not by the Minkowskian formula

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{1.15}$$

but instead by

$$c^2 d\tau^2 = \left(1 + \frac{2\Phi(x,y,z)}{c^2}\right)c^2 dt^2 - \left(1 - \frac{2\Phi(x,y,z)}{c^2}\right)(dx^2 + dy^2 + dz^2), \tag{1.16}$$

where $\Phi/c^2 \ll 1$. Let Alice have spatial position $\mathbf{x}_A = (x_A, y_A, z_A)$ and Bob have spatial position $\mathbf{x}_B$. Assume that Alice sends a light signal to Bob at time t_A and a second signal at time $t_A + \Delta t$. Let Bob receive the first signal at time t_B. What time does he receive the second signal? We haven't discussed how one determines the trajectory of the light ray but this doesn't matter. The above geometry does not depend on t. Hence the trajectory of the second signal must be the same as the first signal (whatever this is) but simply shifted by a time Δt:

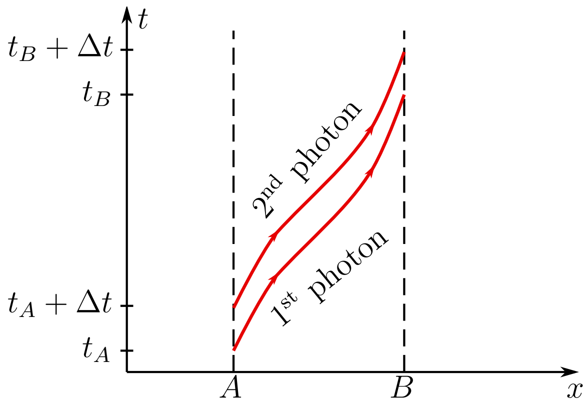


Figure 1.7: Light ray paths

Hence Bob receives the second signal at time $t_B + \Delta t$. The proper time interval between the signals sent by Alice is given by

$$\Delta\tau_A^2 = \left(1 + \frac{2\Phi_A}{c^2}\right)\Delta t^2, \tag{1.17}$$

where $\Phi_A \equiv \Phi(\mathbf{x}_A)$. (Note $\Delta x = \Delta y = \Delta z = 0$ because her signals are sent from the same spatial position.) Hence, using $\Phi/c^2 \ll 1$,

$$\Delta\tau_A = \left(1 + \frac{2\Phi_A}{c^2}\right)^{1/2} \approx \left(1 + \frac{\Phi_A}{c^2}\right)\Delta t. \tag{1.18}$$

Similarly, the proper time between the signals received by Bob is

$$\Delta\tau_B \approx \left(1 + \frac{\Phi_B}{c^2}\right)\Delta t. \tag{1.19}$$

Hence, eliminating Δt:

$$\Delta\tau_B \approx \left(1 + \frac{\Phi_B}{c^2}\right)\left(1 + \frac{\Phi_A}{c^2}\right)^{-1}\Delta\tau_A \approx \left(1 + \frac{\Phi_B - \Phi_A}{c^2}\right)\Delta\tau_A, \tag{1.20}$$

which is just equation (1.14). The difference in the rates of the two clocks has been explained by the geometry of spacetime. The geometry (1.16) is actually the geometry predicted by General Relativity outside a time-independent, non-rotating distribution of matter, at least when gravity is weak, i.e., $|\Phi|/c^2 \ll 1$. (This is true in the Solar System: $|\Phi|/c^2 = GM/(rc^2) \sim 10^{-5}$ at the surface of the Sun.)

Chapter 2

Manifolds and tensors

2.1 Introduction

In Minkowski spacetime we usually use inertial frame coordinates (t, x, y, z) since these are adapted to the symmetries of the spacetime so using these coordinates simplifies the form of physical laws. However, a general spacetime has no symmetries and therefore no preferred set of coordinates. In fact, a single set of coordinates might not be sufficient to describe the spacetime. A simple example of this is provided by spherical polar coordinates (θ, ϕ) on the surface of the unit sphere S^2 in $\mathbb{R}^3$:

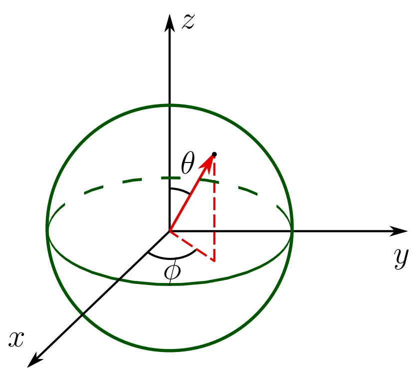


Figure 2.1: Spherical polar coordinates

These coordinates are not well-defined at $\theta = 0, \pi$ (what is the value of ϕ there?). Furthermore, the coordinate ϕ is discontinuous at $\phi = 0$ or 2π.

To describe S^2 so that a pair of coordinates is assigned in a smooth way to every point, we need to use several overlapping sets of coordinates. Generalizing this example leads to the idea of a *manifold*. In GR, we assume that spacetime is a 4-dimensional differentiable manifold.

2.2 Differentiable manifolds

You know how to do calculus on $\mathbb{R}^n$. How do you do calculus on a curved space, e.g., S^2? Locally, S^2 looks like $\mathbb{R}^2$ so one can carry over standard results. However, one has to confront the fact that it is impossible to use a single coordinate system on S^2. In order to do calculus we need our coordinates systems to "mesh together" in a smooth way. Mathematically, this is captured by the notion of a differentiable manifold:

Definition. An n-dimensional *differentiable manifold* is a set M together with a collection of subsets $\mathcal{O}_\alpha$ such that

1. $\bigcup_\alpha \mathcal{O}_\alpha = M$, i.e., the subsets $\mathcal{O}_\alpha$ cover M
2. For each α there is a one-to-one and onto map $\phi_\alpha : \mathcal{O}_\alpha \to \mathcal{U}_\alpha$ where $\mathcal{U}_\alpha$ is an open subset of $\mathbb{R}^n$.
3. If $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ overlap, i.e., $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \emptyset$ then $\phi_\beta \circ \phi_\alpha^{-1}$ maps from $\phi_\alpha(\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \subset \mathcal{U}_\alpha \subset \mathbb{R}^n$ to $\phi_\beta(\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \subset \mathcal{U}_\beta \subset \mathbb{R}^n$. We require that this map be smooth (infinitely differentiable).

The maps ϕ_α are called *charts* or *coordinate systems*. The set $\{\phi_\alpha\}$ is called an *atlas*.

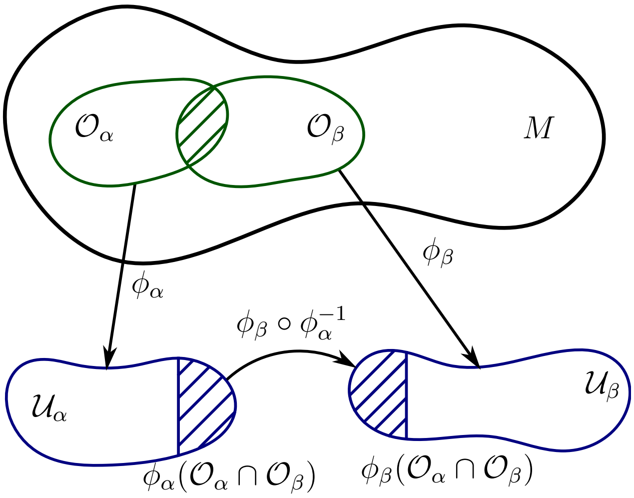


Figure 2.2: Overlapping charts

Remarks.

1. Sometimes we shall write $\phi_\alpha(p) = (x_\alpha^1(p), x_\alpha^2(p), \ldots x_\alpha^n(p))$ and refer to $x_\alpha^i(p)$ as the coordinates of p.

2. Strictly speaking, we have defined above the notion of a *smooth* manifold. If we replace "smooth" in the definition by C^k (k-times continuously differentiable) then we obtain a C^k-manifold. We shall always assume the manifold is smooth.

Examples.

1. $\mathbb{R}^n$: this is a manifold with atlas consisting of the single chart $\phi : (x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^n)$.

2. S^1: the unit circle, i.e., the subset of $\mathbb{R}^2$ given by $(\cos\theta, \sin\theta)$ with $\theta \in \mathbb{R}$. We can't define a chart by using $\theta \in [0, 2\pi)$ as a coordinate because $[0, 2\pi)$ is not open. Instead let P be the point $(1, 0)$ and define one chart by $\phi_1 : S^1 - \{P\} \to (0, 2\pi)$, $\phi_1(p) = \theta_1$ with θ_1 defined by Fig. 2.3.

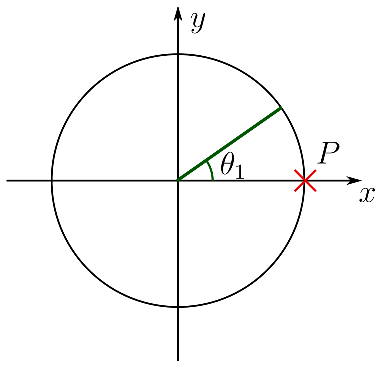


Figure 2.3: Definition of θ_1

Now let Q be the point $(-1, 0)$ and define a second chart by $\phi_2 : S^1 - \{Q\} \to (-\pi, \pi)$, $\phi_2(p) = \theta_2$ where θ_2 is defined by Fig. 2.4.

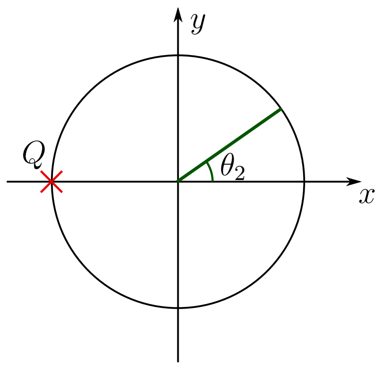


Figure 2.4: Definition of θ_2

Neither chart covers all of S^1 but together they form an atlas. The charts overlap on the "upper" semi-circle and on the "lower" semi-circle. On the first of these we have $\theta_2 = \phi_2 \circ \phi_1^{-1}(\theta_1) = \theta_1$. On the second we have $\theta_2 = \phi_2 \circ \phi_1^{-1}(\theta_1) = \theta_1 - 2\pi$. These are obviously smooth functions.

3. S^2: the two-dimensional sphere defined by the surface $x^2 + y^2 + z^2 = 1$ in Euclidean space. Introduce spherical polar coordinates in the usual way:

$$x = \sin\theta\cos\phi, \qquad y = \sin\theta\sin\phi, \qquad z = \cos\theta \tag{2.1}$$

these equations define $\theta \in (0, \pi)$ and $\phi \in (0, 2\pi)$ uniquely. Hence this defines a chart $\psi : \mathcal{O} \to \mathcal{U}$ where $\mathcal{O}$ is S^2 with the points $(0, 0, \pm 1)$ and the line of longitude $y = 0, x > 0$ removed, see Fig. 2.5, and $\mathcal{U}$ is $(0, \pi) \times (0, 2\pi) \subset \mathbb{R}^2$.

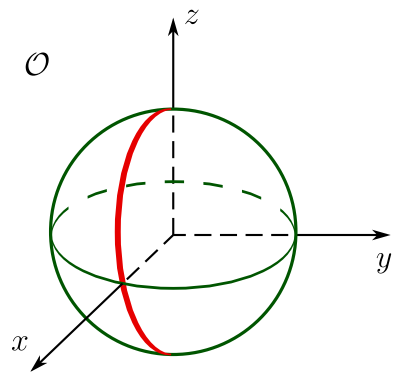


Figure 2.5: The subset $\mathcal{O} \subset S^2$: points with $\theta = 0, \pi$ and $\phi = 0, 2\pi$ are removed.

We can define a second chart using a different set of spherical polar coordinates defined as follows:

$$x = -\sin\theta'\cos\phi', \qquad y = \cos\theta', \qquad z = \sin\theta'\sin\phi', \tag{2.2}$$

where $\theta' \in (0, \pi)$ and $\phi' \in (0, 2\pi)$ are uniquely defined by these equations. This is a chart $\psi' : \mathcal{O}' \to \mathcal{U}'$, where $\mathcal{O}'$ is S^2 with the points $(0, \pm 1, 0)$ and the line $z = 0, x < 0$ removed, see Fig. 2.6, and $\mathcal{U}'$ is $(0, \pi) \times (0, 2\pi)$. Clearly $S^2 = \mathcal{O} \cup \mathcal{O}'$. The functions $\psi \circ \psi'^{-1}$ and $\psi' \circ \psi^{-1}$ are smooth on $\mathcal{O} \cap \mathcal{O}'$ so these two charts define an atlas for S^2.

Remark. A given set M may admit many atlases, e.g., one can simply add extra charts to an atlas. We don't want to regard this as producing a distinct manifold so we make the following definition:

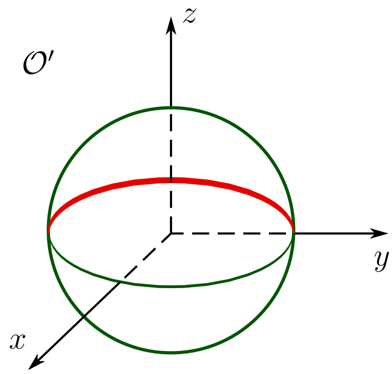


Figure 2.6: The subset $\mathcal{O}' \subset S^2$: points with $\theta' = 0, \pi$ and $\phi' = 0, 2\pi$ are removed.

Definition. Two atlases are *compatible* if their union is also an atlas. The union of all atlases compatible with a given atlas is called a *complete atlas*: it is an atlas which is not contained in any other atlas.

Remark. We will always assume that were are dealing with a complete atlas. (None of the above examples gives a complete atlas; such atlases necessarily contain infinitely many charts.)

2.3 Smooth functions

We will need the notion of a smooth function on a smooth manifold. If $\phi : \mathcal{O} \to \mathcal{U}$ is a chart and $f : M \to \mathbb{R}$ then note that $f \circ \phi^{-1}$ is a map from $\mathcal{U}$, i.e., a subset of $\mathbb{R}^n$, to $\mathbb{R}$.

Definition. A function $f : M \to \mathbb{R}$ is *smooth* if, and only if, for any chart ϕ, $F \equiv f \circ \phi^{-1} : \mathcal{U} \to \mathbb{R}$ is a smooth function.

Remark. In GR, a function $f : M \to \mathbb{R}$ is sometimes called a *scalar field.*

Examples.

1. Consider the example of S^1 discussed above. Let $f : S^1 \to \mathbb{R}$ be defined by $f(x, y) = x$ where (x, y) are the Cartesian coordinates in $\mathbb{R}^2$ labelling a point on S^1. In the first chart ϕ_1 we have $f \circ \phi_1^{-1}(\theta_1) = f(\cos\theta_1, \sin\theta_1) = \cos\theta_1$, which is smooth. Similary $f \circ \phi_2^{-1}(\theta_2) = \cos\theta_2$ is also smooth. If ϕ is any other chart then we can write $f \circ \phi^{-1} = (f \circ \phi_i^{-1}) \circ (\phi_i \circ \phi^{-1})$, which is smooth because we've just seen that $f \circ \phi_i^{-1}$ are smooth, and $\phi_i \circ \phi^{-1}$ is smooth from the definition of a manifold. Hence f is a smooth function.

2. Consider a manifold M with a chart $\phi : \mathcal{O} \to \mathcal{U} \subset \mathbb{R}^n$. Denote the other charts in the atlas by ϕ_α. Let $\phi : p \mapsto (x^1(p), x^2(p), \ldots x^n(p))$. Then we can

regard x^1 (say) as a function on the subset $\mathcal{O}$ of M. Is it a smooth function? Yes: $x^1 \circ \phi_\alpha^{-1}$ is smooth for any chart ϕ_α, because it is the first component of the map $\phi \circ \phi_\alpha^{-1}$, and the latter is smooth by the definition of a manifold.

3. Often it is convenient to define a function by specifying F instead of f. More precisely, given an atlas $\{\phi_\alpha\}$, we define f by specifying functions $F_\alpha : \mathcal{U}_\alpha \to \mathbb{R}$ and then setting $f = F_\alpha \circ \phi_\alpha$. One has to make sure that the resulting definition is independent of α on chart overlaps. For example, for S^1 using the atlas discussed above, define $F_1 : (0, 2\pi) \to \mathbb{R}$ by $\theta_1 \mapsto \sin(m\theta_1)$ and $F_2 : (-\pi, \pi) \to \mathbb{R}$ by $\theta_2 \mapsto \sin(m\theta_2)$, where m is an integer. On the chart overlaps we have $F_1 \circ \phi_1 = F_2 \circ \phi_2$ because θ_1 and θ_2 differ by a multiple of 2π on both overlaps. Hence this defines a function on S^1.

Remark. After a while we will stop distinguishing between f and F, i.e., we will say $f(x)$ when we mean $F(x)$.

2.4 Curves and vectors

$\mathbb{R}^n$, or Minkowski spacetime, has the structure of a vector space, e.g., it makes sense to add the position vectors of points. One can view more general vectors, e.g., the 4-velocity of a particle, as vectors in the space itself. This structure does not extend to more general manifolds, e.g., S^2. So we need to discuss how to define vectors on manifolds.

For a surface in $\mathbb{R}^3$, the set of all vectors tangent to the surface at some point p defines the *tangent plane* to the surface at p (see Fig. 2.7). This has the structure of a 2d vector space. Note that the tangent planes at two different points p and q are *different*. It does not make sense to compare a vector at p with a vector at q. For example: if one tried to define the sum of a vector at p and a vector at q then to which tangent plane would the sum belong?

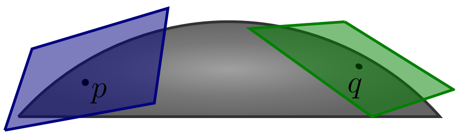


Figure 2.7: Tangent planes.

On a surface, the tangent vector to a curve in the surface is automatically tangent to the surface. We take this as our starting point for defining vectors on a general manifold. We start by defining the notion of a curve in a manifold, and then the notion of a tangent vector to a curve at a point p. We then show that the set of all such tangent vectors at p forms a vector space $T_p(M)$. This is the analogue of the tangent plane to a surface but it makes no reference to any embedding into a higher-dimensional space.

Definition A smooth curve in a differentiable manifold M is a smooth function $\lambda : I \to M$, where I is an open interval in $\mathbb{R}$ (e.g. $(0,1)$ or $(-1,\infty)$). By this we mean that $\phi_\alpha \circ \lambda$ is a smooth map from I to $\mathbb{R}^n$ for all charts ϕ_α.

Let $f : M \to \mathbb{R}$ and $\lambda : I \to M$ be a smooth function and a smooth curve respectively. Then $f \circ \lambda$ is a map from I to $\mathbb{R}$. Hence we can take its derivative to obtain the rate of change of f along the curve:

$$\frac{d}{dt}\left[(f \circ \lambda)(t)\right] = \frac{d}{dt}\left[f(\lambda(t))\right] \tag{2.3}$$

In $\mathbb{R}^n$ we are used to the idea that the rate of change of f along the curve at a point p is given by the directional derivative $\mathbf{X}_p \cdot (\nabla f)_p$ where $\mathbf{X}_p$ is the tangent to the curve at p. Note that the vector $\mathbf{X}_p$ defines a linear map from the space of smooth functions on $\mathbb{R}^n$ to $\mathbb{R}$: $f \mapsto \mathbf{X}_p \cdot (\nabla f)_p$. This is how we define a tangent vector to a curve in a general manifold:

Definition. Let $\lambda : I \to M$ be a smooth curve with (wlog) $\lambda(0) = p$. The *tangent vector to* λ *at* p is the linear map X_p from the space of smooth functions on M to $\mathbb{R}$ defined by

$$X_p(f) = \left\{\frac{d}{dt}\left[f(\lambda(t))\right]\right\}_{t=0} \tag{2.4}$$

Note that this satisfies two important properties: (i) it is linear, i.e., $X_p(f+g) = X_p(f)+X_p(g)$ and $X_p(\alpha f) = \alpha X_p(f)$ for any constant α; (ii) it satisfies the Leibniz rule $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$, where f and g are smooth functions and fg is their product.

If $\phi = (x^1, x^2, \dots x^n)$ is a chart defined in a neighbourhood of p and $F \equiv f \circ \phi^{-1}$ then we have $f \circ \lambda = f \circ \phi^{-1} \circ \phi \circ \lambda = F \circ \phi \circ \lambda$ and hence

$$X_p(f) = \left(\frac{\partial F(x)}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{dx^\mu(\lambda(t))}{dt}\right)_{t=0} \tag{2.5}$$

Note that (i) the first term on the RHS depends only on f and ϕ, and the second term on the RHS depends only on ϕ and λ; (ii) we are using the Einstein summation convention, i.e., μ is summed from 1 to n in the above expression.

Proposition. The set of all tangent vectors at p forms a n-dimensional vector space, the *tangent space* $T_p(M)$.

Proof. Consider curves λ and κ through p, wlog $\lambda(0) = \kappa(0) = p$. Let their tangent vectors at p be X_p and Y_p respectively. We need to define addition of tangent vectors and multiplication by a constant. let α and β be constants. We define $\alpha X_p + \beta Y_p$ to be the linear map $f \mapsto \alpha X_p(f) + \beta Y_p(f)$. Next we need to show that this linear map is indeed the tangent vector to a curve through p. Let $\phi = (x^1, \ldots, x^n)$ be a chart defined in a neighbourhood of p. Consider the following curve:

$$\nu(t) = \phi^{-1}\left[\alpha(\phi(\lambda(t)) - \phi(p)) + \beta(\phi(\kappa(t)) - \phi(p)) + \phi(p)\right] \tag{2.6}$$

Note that $\nu(0) = p$. Let Z_p denote the tangent vector to this curve at p. From equation (2.5) we have

$$\begin{aligned} Z_p(f) &= \left(\frac{\partial F(x)}{\partial x^\mu}\right)_{\phi(p)} \left\{\frac{d}{dt}\left[\alpha(x^\mu(\lambda(t)) - x^\mu(p)) + \beta(x^\mu(\kappa(t)) - x^\mu(p)) + x^\mu(p)\right]\right\}_{t=0} \\ &= \left(\frac{\partial F(x)}{\partial x^\mu}\right)_{\phi(p)} \left[\alpha\left(\frac{dx^\mu(\lambda(t))}{dt}\right)_{t=0} + \beta\left(\frac{dx^\mu(\kappa(t))}{dt}\right)_{t=0}\right] \\ &= \alpha X_p(f) + \beta Y_p(f) \\ &= (\alpha X_p + \beta Y_p)(f). \end{aligned}$$

Since this is true for any smooth function f, we have $Z_p = \alpha X_p + \beta Y_p$ as required. Hence $\alpha X_p + \beta Y_p$ is tangent to the curve ν at p. It follows that the set of tangent vectors at p forms a vector space (the zero vector is realized by the curve $\lambda(t) = p$ for all t).

The next step is to show that this vector space is n-dimensional. To do this, we exhibit a basis. Let $1 \leq \mu \leq n$. Consider the curve λ_μ through p defined by

$$\lambda_\mu(t) = \phi^{-1}(x^1(p), \ldots, x^{\mu-1}(p), x^\mu(p) + t, x^{\mu+1}(p), \ldots, x^n(p)). \tag{2.7}$$

The tangent vector to this curve at p is denoted $\left(\frac{\partial}{\partial x^\mu}\right)_p$. To see why, note that, using equation (2.5)

$$\left(\frac{\partial}{\partial x^\mu}\right)_p (f) = \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)}. \tag{2.8}$$

The n tangent vectors $\left(\frac{\partial}{\partial x^\mu}\right)_p$ are linearly independent. To see why, assume that there exist constants α^μ such that $\alpha^\mu \left(\frac{\partial}{\partial x^\mu}\right)_p = 0$. Then, for any function f we must have

$$\alpha^\mu \left(\frac{\partial F(x)}{\partial x^\mu}\right)_{\phi(p)} = 0. \tag{2.9}$$

Choosing $F = x^\nu$, this reduces to $\alpha^\nu = 0$. Letting this run over all values of ν we see that all of the constants α^ν must vanish, which proves linear independence.

Finally we must prove that these tangent vectors span the vector space. This follows from equation (2.5), which can be rewritten

$$X_p(f) = \left(\frac{dx^\mu(\lambda(t))}{dt}\right)_{t=0} \left(\frac{\partial}{\partial x^\mu}\right)_p (f) \tag{2.10}$$

this is true for any f hence

$$X_p = \left(\frac{dx^\mu(\lambda(t))}{dt}\right)_{t=0} \left(\frac{\partial}{\partial x^\mu}\right)_p, \tag{2.11}$$

i.e. X_p can be written as a linear combination of the n tangent vectors $\left(\frac{\partial}{\partial x^\mu}\right)_p$. These n vectors therefore form a basis for $T_p(M)$, which establishes that the tangent space is n-dimensional. QED.

Remark. The basis $\{\left(\frac{\partial}{\partial x^\mu}\right)_p, \mu = 1, \ldots n\}$ is chart-dependent: we had to choose a chart ϕ defined in a neighbourhood of p to define it. Choosing a different chart would give a different basis for $T_p(M)$. A basis defined this way is sometimes called a *coordinate basis.*

Definition. Let $\{e_\mu, \mu = 1 \ldots n\}$ be a basis for $T_p(M)$ (not necessarily a coordinate basis). We can expand any vector $X \in T_p(M)$ as $X = X^\mu e_\mu$. We call the numbers X^μ the *components* of X with respect to this basis.

Example. Using the coordinate basis $e_\mu = (\partial/\partial x^\mu)_p$, equation (2.11) shows that the tangent vector X_p to a curve $\lambda(t)$ at p (where $t = 0$) has components

$$X_p^\mu = \left(\frac{dx^\mu(\lambda(t))}{dt}\right)_{t=0}. \tag{2.12}$$

Remark. Note the placement of indices. We shall sum over repeated indices if one such index appears "upstairs" (as a superscript, e.g., X^μ) and the other "downstairs" (as a subscript, e.g., e_μ). (The index μ on $\left(\frac{\partial}{\partial x^\mu}\right)_p$ is regarded as downstairs.) If an equation involves the same index more than twice, or twice but both times upstairs or both times downstairs (e.g. $X_\mu Y_\mu$) then a mistake has been made.

Let's consider the relationship between different coordinate bases. Let $\phi = (x^1, \ldots, x^n)$ and $\phi' = (x'^1, \ldots, x'^n)$ be two charts defined in a neighbourhood of p. Then, for any smooth function f, we have

$$\begin{aligned}\left(\frac{\partial}{\partial x^\mu}\right)_p (f) &= \left(\frac{\partial}{\partial x^\mu}(f \circ \phi^{-1})\right)_{\phi(p)} \\ &= \left(\frac{\partial}{\partial x^\mu}[(f \circ \phi'^{-1}) \circ (\phi' \circ \phi^{-1})]\right)_{\phi(p)}\end{aligned}$$

Now let $F' = f \circ \phi'^{-1}$. This is a function of the coordinates x'. Note that the components of $\phi' \circ \phi^{-1}$ are simply the functions $x'^{\mu}(x)$, i.e., the primed coordinates expressed in terms of the unprimed coordinates. Hence what we have is easy to evaluate using the chain rule:

$$\begin{aligned}
\left(\frac{\partial}{\partial x^\mu}\right)_p (f) &= \left(\frac{\partial}{\partial x^\mu}(F'(x'(x)))\right)_{\phi(p)} \\
&= \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial F'(x')}{\partial x'^\nu}\right)_{\phi'(p)} \\
&= \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial}{\partial x'^\nu}\right)_p (f)
\end{aligned}$$

Hence we have

$$\left(\frac{\partial}{\partial x^\mu}\right)_p = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial}{\partial x'^\nu}\right)_p \tag{2.13}$$

This expresses one set of basis vectors in terms of the other. Let X^μ and X'^μ denote the components of a vector with respect to the two bases. Then we have

$$X = X^\nu \left(\frac{\partial}{\partial x^\nu}\right)_p = X^\nu \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)_{\phi(p)} \left(\frac{\partial}{\partial x'^\mu}\right)_p \tag{2.14}$$

and hence

$$X'^\mu = X^\nu \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)_{\phi(p)} \tag{2.15}$$

Elementary treatments of GR usually *define* a vector to be a set of numbers $\{X^\mu\}$ that transforms according to this rule under a change of coordinates. More precisely, they usually call this a "contravariant vector".

2.5 Covectors

Recall the following from linear algebra:

Definition. Let V be a real vector space. The *dual space* V^* of V is the vector space of linear maps from V to $\mathbb{R}$.

Lemma. If V is n-dimensional then so is V^*. If $\{e_\mu,\, \mu = 1, \ldots, n\}$ is a basis for V then V^* has a basis $\{f^\mu,\, \mu = 1, \ldots, n\}$, the *dual basis* defined by $f^\mu(e_\nu) = \delta^\mu_\nu$ (if $X = X^\mu e_\mu$ then $f^\mu(X) = X^\nu f^\mu(e_\nu) = X^\mu$).

Since V and V^* have the same dimension, they are isomorphic. For example the linear map defined by $e_\mu \mapsto f^\mu$ is an isomorphism. But this is basis-dependent: a different choice of basis would give a different isomorphism. In contrast, there is a *natural* (basis-independent) isomorphism between V and $(V^*)^*$:

Theorem. If V is finite dimensional then $(V^*)^*$ is naturally isomorphic to V. The isomorphism is $\Phi : V \to (V^*)^*$ where $\Phi(X)(\omega) = \omega(X)$ for all $\omega \in V^*$.

Now we return to manifolds:

Definition. The dual space of $T_p(M)$ is denoted $T_p^*(M)$ and called the *cotangent space* at p. An element of this space is called a *covector (or 1-form) at* p. If $\{e_\mu\}$ is a basis for $T_p(M)$ and $\{f^\mu\}$ is the dual basis then we can expand a covector η as $\eta_\mu f^\mu$. η_μ are called the components of η.

Note that (i) $\eta(e_\mu) = \eta_\nu f^\nu(e_\mu) = \eta_\mu$; (ii) if $X \in T_p(M)$ then $\eta(X) = \eta(X^\mu e_\mu) = X^\mu \eta(e_\mu) = X^\mu \eta_\mu$ (note the placement of indices!)

Definition. Let $f : M \to \mathbb{R}$ be a smooth function. Define a covector $(df)_p$ by $(df)_p(X) = X(f)$ for any vector $X \in T_p(M)$. $(df)_p$ is the *gradient of* f *at* p.

Examples.

1. Let $(x^1, \ldots, x^n)$ be a coordinate chart defined in a neighbourhood of p, recall that x^μ is a smooth function (in this neighbourhood) so we can take $f = x^\mu$ in the above definition to define n covectors $(dx^\mu)_p$. Note that

$$(dx^\mu)_p \left(\left(\frac{\partial}{\partial x^\nu} \right)_p \right) = \left(\frac{\partial x^\mu}{\partial x^\nu} \right)_p = \delta^\mu_\nu \tag{2.16}$$

   Hence $\{(dx^\mu)_p\}$ is the dual basis of $\{(\partial/\partial x^\mu)_p\}$.

2. To explain why we call $(df)_p$ the gradient of f at p, observe that its components in a coordinate basis are

$$[(df)_p]_\mu = (df)_p \left(\left(\frac{\partial}{\partial x^\mu} \right)_p \right) = \left(\frac{\partial}{\partial x^\mu} \right)_p (f) = \left(\frac{\partial F}{\partial x^\mu} \right)_{\phi(p)} \tag{2.17}$$

   where the first equality uses (i) above, the second equality is the definition of $(df)_p$ and the final equality used (2.8).

Exercise. Consider two different charts $\phi = (x^1, \ldots, x^n)$ and $\phi' = (x'^1, \ldots, x'^n)$ defined in a neighbourhood of p. Show that

$$(dx^\mu)_p = \left(\frac{\partial x^\mu}{\partial x'^\nu} \right)_{\phi'(p)} (dx'^\nu)_p, \tag{2.18}$$

and hence that, if ω_μ and ω'_μ are the components of $\omega \in T_p^*(M)$ w.r.t. the two coordinate bases, then

$$\omega'_\mu = \left(\frac{\partial x^\nu}{\partial x'^\mu}\right)_{\phi'(p)} \omega_\nu. \tag{2.19}$$

Elementary treatements of GR take this as the *definition* of a covector, which they usually call a "covariant vector".

2.6 Abstract index notation

So far, we have used Greek letters $\mu, \nu, \ldots$ to denote components of vectors or covectors with respect to a basis, and also to label the basis vectors themselves (e.g. e_μ). Equations involving such indices are assumed to hold *only in that basis*. For example an equation of the form $X^\mu = \delta_1^\mu$ says that, in a particular basis, a vector X has only a single non-vanishing component. This will not be true in other bases. Furthermore, if we were just presented with this equation, we would not even know whether or not the quantities $\{X^\mu\}$ are the components of a vector or just a set of n numbers.

The abstract index notation uses Latin letters $a, b, c, \ldots$. A vector X is denoted X^a or X^b or X^c etc. The letter used in the superscript does not matter. What matters is that there *is* a superscript Latin letter. This tells us that the object in question is a vector. We emphasize: X^a represents the vector itself, *not* a component of the vector. Similarly we denote a covector η by η_a (or η_b etc).

If we have an equation involving abstract indices then we can obtain an equation valid in any particular basis simply by replacing the abstract indices by basis indices (e.g. $a \to \mu$, $b \to \nu$ etc.). For example, consider the quantity $\eta_a X^a$ in the abstract index notation. We see that this involves a covector η_a and a vector X^a. Furthermore, in any basis, this quantity is equal to $\eta_\mu X^\mu = \eta(X)$. Hence $\eta_a X^a$ is the abstract index way of writing $\eta(X)$. Similarly, if f is a smooth function then $X(f) = X^a(df)_a$.

Conversely, if one has an equation involving Greek indices but one knows that it is true for an *arbitrary* basis then one can replace the Greek indices with Latin letters.

Latin indices must respect the rules of the summation convention so equations of the form $\eta_a \eta_a = 1$ or $\eta_b = 2$ do not make sense.

2.7 Tensors

In Newtonian physics, you are familiar with the idea that certain physical quantities are described by tensors (e.g. the inertia tensor). You may have encountered the idea that the Maxwell field in special relativity is described by a tensor. Tensors are very important in GR because the curvature of spacetime is described with tensors. In this section we shall define tensors at a point p and explain some of their basic properties.

Definition. A tensor of type (r, s) at p is a multilinear map

$$T : T_p^*(M) \times \ldots \times T_p^*(M) \times T_p(M) \times \ldots \times T_p(M) \to \mathbb{R}. \tag{2.20}$$

where there are r factors of $T_p^*(M)$ and s factors of $T_p(M)$. (Multilinear means that the map is linear in each argument.)

In other words, given r covectors and s vectors, a tensor of type (r, s) produces a real number.

Examples.

1. A tensor of type $(0, 1)$ is a linear map $T_p(M) \to \mathbb{R}$, i.e., it is a covector.
2. A tensor of type $(1, 0)$ is a linear map $T_p^*(M) \to \mathbb{R}$, i.e., it is an element of $(T_p^*(M))^*$ but this is naturally isomorphic to $T_p(M)$ hence a tensor of type $(1, 0)$ is a vector. To see how this works, given a vector $X \in T_p(M)$ we define a linear map $T_p^*(M) \to \mathbb{R}$ by $\eta \mapsto \eta(X)$ for any $\eta \in T_p^*(M)$.
3. We can define a $(1, 1)$ tensor δ by $\delta(\omega, X) = \omega(X)$ for any covector ω and vector X.

Definition. Let T be a tensor of type (r, s) at p. If $\{e_\mu\}$ is a basis for $T_p(M)$ with dual basis $\{f^\mu\}$ then the *components* of T in this basis are the numbers

$$T^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s} = T(f^{\mu_1}, f^{\mu_2}, \ldots, f^{\mu_r}, e_{\nu_1}, e_{\nu_2}, \ldots, e_{\nu_s}) \tag{2.21}$$

In the abstract index notation, we denote T by $T^{a_1a_2\ldots a_r}{}_{b_1b_2\ldots b_s}$.

Remark. Tensors of type (r, s) at p can be added together and multiplied by a constant, hence they form a vector space. Since such a tensor has n^{r+s} components, it is clear that this vector space has dimension n^{r+s}.

Examples.

1. Consider the tensor δ defined above. Its components are

   $$\delta^\mu{}_\nu = \delta(f^\mu, e_\nu) = f^\mu(e_\nu) = \delta^\mu_\nu, \tag{2.22}$$

   where the RHS is a Kronecker delta. This is true in any basis, so in the abstract index notation we write δ as δ^a_b.

2. Consider a $(2,1)$ tensor. Let η and ω be covectors and X a vector. Then in our basis we have

$$T(\eta,\omega,X) = T(\eta_\mu f^\mu, \omega_\nu f^\nu, X^\rho e_\rho) = \eta_\mu \omega_\nu X^\rho T(f^\mu, f^\nu, e_\rho) = T^{\mu\nu}{}_\rho \eta_\mu \omega_\nu X^\rho \tag{2.23}$$

Now the basis we chose was *arbitrary*, hence we can immediately convert this to a basis-independent equation using the abstract index notation:

$$T(\eta,\omega,X) = T^{ab}{}_c \eta_a \omega_b X^c. \tag{2.24}$$

This formula generalizes in the obvious way to a (r,s) tensor.

We have discussed the transformation of vectors and covectors components under a change of *coordinate* basis. Let's now examine how tensor components transform under an *arbitrary* change of basis. Let $\{e_\mu\}$ and $\{e'_\mu\}$ be two bases for $T_p(M)$. Let $\{f^\mu\}$ and $\{f'^\mu\}$ denote the corresponding dual bases. Expanding the primed bases in terms of the unprimed bases gives

$$f'^\mu = A^\mu{}_\nu f^\nu, \qquad e'_\mu = B^\nu{}_\mu e_\nu \tag{2.25}$$

for some matrices $A^\mu{}_\nu$ and $B^\nu{}_\mu$. These matrices are related because:

$$\delta^\mu_\nu = f'^\mu(e'_\nu) = A^\mu{}_\rho f^\rho (B^\sigma{}_\nu e_\sigma) = A^\mu{}_\rho B^\sigma{}_\nu f^\rho(e_\sigma) = A^\mu{}_\rho B^\sigma{}_\nu \delta^\rho_\sigma = A^\mu{}_\rho B^\rho{}_\nu. \tag{2.26}$$

Hence $B^\mu{}_\nu = (A^{-1})^\mu{}_\nu$. For a change between coordinate bases, our previous results give

$$A^\mu{}_\nu = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right), \qquad B^\mu{}_\nu = \left(\frac{\partial x^\mu}{\partial x'^\nu}\right) \tag{2.27}$$

and these matrices are indeed inverses of each other (from the chain rule).

Exercise. Show that under an arbitrary change of basis, the components of a vector X and a covector η transform as

$$X'^\mu = A^\mu{}_\nu X^\nu, \qquad \eta'{}_\mu = (A^{-1})^\nu{}_\mu \eta_\nu. \tag{2.28}$$

Show that the components of a $(2,1)$ tensor T transform as

$$T'^{\mu\nu}{}_\rho = A^\mu{}_\sigma A^\nu{}_\tau (A^{-1})^\lambda{}_\rho T^{\sigma\tau}{}_\lambda. \tag{2.29}$$

The corresponding result for a (r,s) tensor is an obvious generalization of this formula.

Given a (r,s) tensor, we can construct a $(r-1,s-1)$ tensor by *contraction*. This is easiest to demonstrate with an example.

Example. Let T be a tensor of type $(3,2)$. Define a new tensor S of type $(2,1)$ as follows

$$S(\omega,\eta,X) = T(f^\mu,\omega,\eta,e_\mu,X) \tag{2.30}$$

where $\{e_\mu\}$ is a basis and $\{f^\mu\}$ is the dual basis, ω and η are arbitrary covectors and X is an arbitrary vector. This definition is basis-independent because

$$\begin{aligned} T(f'^\mu,\omega,\eta,e'_\mu,X) &= T(A^\mu{}_\nu f^\nu,\omega,\eta,(A^{-1})^\rho{}_\mu e_\rho,X) \\ &= (A^{-1})^\rho{}_\mu A^\mu{}_\nu T(f^\nu,\omega,\eta,e_\rho,X) \\ &= T(f^\mu,\omega,\eta,e_\mu,X). \end{aligned}$$

The components of S and T are related by $S^{\mu\nu}{}_\rho = T^{\sigma\mu\nu}{}_{\sigma\rho}$ in any basis. Since this is true in any basis, we can write it using the abstract index notation as

$$S^{ab}{}_c = T^{dab}{}_{dc} \tag{2.31}$$

Note that there are other $(2,1)$ tensors that we can build from $T^{abc}{}_{de}$. For example, there is $T^{abd}{}_{cd}$, which corresponds to replacing the RHS of (2.30) with $T(\omega,\eta,f^\mu,X,e_\mu)$. The abstract index notation makes it clear how many different tensors can be defined this way: we can define a new tensor by "contracting" any upstairs index with any downstairs index.

Another important way of constructing new tensors is by taking the product of two tensors:

Definition. If S is a tensor of type (p,q) and T is a tensor of type (r,s) then the *outer product* of S and T, denoted $S\otimes T$ is a tensor of type $(p+r,q+s)$ defined by

$$\begin{aligned} (S &\otimes T)(\omega_1,\ldots,\omega_p,\eta_1,\ldots,\eta_r,X_1,\ldots,X_q,Y_1,\ldots,Y_s) \\ &= S(\omega_1,\ldots,\omega_p,X_1,\ldots,X_q)T(\eta_1,\ldots,\eta_r,Y_1,\ldots,Y_s) \end{aligned} \tag{2.32}$$

where $\omega_1,\ldots,\omega_p$ and $\eta_1,\ldots,\eta_r$ are arbitrary covectors and $X_1,\ldots,X_q$ and $Y_1,\ldots,Y_s$ are arbitrary vectors.

Exercise. Show that this definition is equivalent to

$$(S\otimes T)^{a_1\ldots a_p b_1\ldots b_r}{}_{c_1\ldots c_q d_1\ldots d_s} = S^{a_1\ldots a_p}{}_{c_1\ldots c_q} T^{b_1\ldots b_r}{}_{d_1\ldots d_s} \tag{2.33}$$

Exercise. Show that, in a coordinate basis, any $(2,1)$ tensor T at p can be written as

$$T = T^{\mu\nu}{}_\rho \left(\frac{\partial}{\partial x^\mu}\right)_p \otimes \left(\frac{\partial}{\partial x^\nu}\right)_p \otimes (dx^\rho)_p \tag{2.34}$$

This generalizes in the obvious way to a (r,s) tensor.

Remark. You may be wondering why we write $T^{ab}{}_c$ instead of T^{ab}_c. At the moment there is no reason why we should not adopt the latter notation. However, it is convenient to generalize our definition of tensors slightly. We have defined a (r, s) tensor to be a linear map with $r + s$ arguments, where the first r arguments are covectors and the final s arguments are vectors. We can generalize this by allowing the covectors and vectors to appear in any order. For example, consider a $(1, 1)$ tensor. This is a map $T_p^*(M) \times T_p(M) \to \mathbb{R}$. But we could just as well have defined it to be a map $T_p(M) \times T_p^*(M) \to \mathbb{R}$. The abstract index notation allows us to distinguish these possibilities easily: the first would be written as $T^a{}_b$ and the second as $T_a{}^b$. $(2, 1)$ tensors come in 3 different types: $T^{ab}{}_c$, $T^a{}_b{}^c$ and $T_a{}^{bc}$. Each type of of (r, s) tensor gives a vector space of dimension n^{r+s} but these vector spaces are naturally isomorphic so often one does not bother to distinguish between them.

There is a final type of tensor operation that we shall need: symmetrization and antisymmetrization. Consider a $(0, 2)$ tensor T. We can define two other $(0, 2)$ tensors S and A as follows:

$$S(X,Y) = \frac{1}{2}(T(X,Y) + T(Y,X)),\ A(X,Y) = \frac{1}{2}(T(X,Y) - T(Y,X)), \quad (2.35)$$

where X and Y are vectors at p. In abstract index notation:

$$S_{ab} = \frac{1}{2}(T_{ab} + T_{ba}), \qquad A_{ab} = \frac{1}{2}(T_{ab} - T_{ba}). \quad (2.36)$$

In a basis, we can regard the components of T as a square matrix. The components of S and A are just the symmetric and antisymmetric parts of this matrix. It is convenient to introduce some notation to describe the operations we have just defined: we write

$$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba}), \qquad T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba}). \quad (2.37)$$

These operations can be applied to more general tensors. For example,

$$T^{(ab)c}{}_d = \frac{1}{2}(T^{abc}{}_d + T^{bac}{}_d). \quad (2.38)$$

We can also symmetrize or antisymmetrize on more than 2 indices. To symmetrize on n indices, we sum over all permutations of these indices and divide the result by $n!$ (the number of permutations). To antisymmetrize we do the same but we weight each term in the sum by the sign of the permutation. The indices that we symmetrize over must be either upstairs or downstairs, they cannot be a mixture. For example,

$$T^{(abc)d} = \frac{1}{3!}\left(T^{abcd} + T^{bcad} + T^{cabd} + T^{bacd} + T^{cbad} + T^{acbd}\right). \quad (2.39)$$

$$T^a{}_{[bcd]} = \frac{1}{3!}\left(T^a{}_{bcd} + T^a{}_{cdb} + T^a{}_{dbc} - T^a{}_{cbd} - T^a{}_{dcb} - T^a{}_{bdc}\right). \tag{2.40}$$

Sometimes we might wish to (anti)symmetrize over indices which are not adjacent. In this case, we use vertical bars to denote indices excluded from the (anti)symmetrization. For example,

$$T_{(a|bc|d)} = \frac{1}{2}\left(T_{abcd} + T_{dbca}\right). \tag{2.41}$$

Exercise. Show that $T^{(ab)}X_{[a|cd|b]} = 0$.

2.8 Tensor fields

So far, we have defined vectors, covectors and tensors at a single point p. However, in physics we shall need to consider how these objects vary in spacetime. This leads us to define vector, covector and tensor *fields*.

Definition. A vector field is a map X which maps any point $p \in M$ to a vector X_p at p. Given a vector field X and a function f we can define a new function $X(f) : M \to \mathbb{R}$ by $X(f) : p \mapsto X_p(f)$. The vector field X is *smooth* if this map is a smooth function for any smooth f.

Example. Given any coordinate chart $\phi = (x^1, \ldots, x^n)$, the vector field $\frac{\partial}{\partial x^\mu}$ is defined by $p \mapsto \left(\frac{\partial}{\partial x^\mu}\right)_p$. Hence

$$\left(\frac{\partial}{\partial x^\mu}\right)(f) : p \mapsto \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)}, \tag{2.42}$$

where $F \equiv f \circ \phi^{-1}$. You should convince yourself that smoothness of f implies that the above map defines a smooth function. Therefore $\partial/\partial x^\mu$ is a smooth vector field. (Note that $(\partial/\partial x^\mu)$ usually won't be defined on the whole manifold M since the chart ϕ might not cover the whole manifold. So strictly speaking this is not a vector field on M but only on a subset of M. We shan't worry too much about this distinction.)

Remark. Since the vector fields $(\partial/\partial x^\mu)_p$ provide a basis for $T_p(M)$ at any point p, we can expand an arbitrary vector field as

$$X = X^\mu \left(\frac{\partial}{\partial x^\mu}\right) \tag{2.43}$$

Since $\partial/\partial x^\mu$ is smooth, it follows that X is smooth if, and only if, its components X^μ are smooth functions.

Definition. A covector field is a map ω which maps any point $p \in M$ to a covector ω_p at p. Given a covector field and a vector field X we can define a function $\omega(X) : M \to \mathbb{R}$ by $\omega(X) : p \mapsto \omega_p(X_p)$. The covector field ω is *smooth* if this function is smooth for any smooth vector field X.

Example. Let f be a smooth function. We have defined $(df)_p$ above. Now we simply let p vary to define a covector field df. Let X be a smooth vector field and f a smooth function. Then $df(X) = X(f)$. This is a smooth function of p (because X is smooth). Hence df is a smooth covector field: the *gradient of f*.

Remark. Taking $f = x^\mu$ reveals that dx^μ is a smooth covector field.

Definition. A (r, s) tensor field is a map T which maps any point $p \in M$ to a (r, s) tensor T_p at p. Given r covector fields $\eta_1, \ldots, \eta_r$ and s vector fields $X_1, \ldots, X_s$ we can define a function $T(\eta_1, \ldots, \eta_r, X_1, \ldots, X_s) : M \to \mathbb{R}$ by $p \mapsto T_p((\eta_1)_p, \ldots, (\eta_r)_p, (X_1)_p, \ldots, (X_s)_p)$. The tensor field T is *smooth* if this function is smooth for any smooth covector fields $\eta_1, \ldots, \eta_r$ and vector fields $X_1, \ldots, X_r$.

Exercise. Show that a tensor field is smooth if, and only if, its components in a coordinate chart are smooth functions.

Remark. Henceforth we shall assume that all tensor fields that we encounter are smooth.

2.9 The commutator

Let X and Y be vector fields and f a smooth function. Since $Y(f)$ is a smooth function, we can act on it with X to form a new smooth function $X(Y(f))$. Does the map $f \mapsto X(Y(f))$ define a vector field? No, because $X(Y(fg)) = X(fY(g) + gY(f)) = fX(Y(g)) + gX(Y(f)) + X(f)Y(g) + X(g)Y(f)$ so the Leibniz law is not satisfied. However, we can also define $Y(X(f))$ and the combination $X(Y(f)) - Y(X(f))$ *does* obey the Leibniz law (check!).

Definition. The *commutator* of two vector fields X and Y is the vector field $[X, Y]$ defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \tag{2.44}$$

for any smooth function f.

To see that this does indeed define a vector field, we can evaluate it in a coordinate chart:

$$\begin{aligned} [X, Y](f) &= X\left(Y^\nu \frac{\partial F}{\partial x^\nu}\right) - Y\left(X^\mu \frac{\partial F}{\partial x^\mu}\right) \\ &= X^\mu \frac{\partial}{\partial x^\mu}\left(Y^\nu \frac{\partial F}{\partial x^\nu}\right) - Y^\nu \frac{\partial}{\partial x^\nu}\left(X^\mu \frac{\partial F}{\partial x^\mu}\right) \end{aligned}$$

$$
\begin{aligned}
&= X^\mu \frac{\partial Y^\nu}{\partial x^\mu}\frac{\partial F}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu}\frac{\partial F}{\partial x^\mu} \\
&= \left(X^\nu \frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu} \right) \frac{\partial F}{\partial x^\mu} \\
&= [X,Y]^\mu \left(\frac{\partial}{\partial x^\mu} \right)(f)
\end{aligned}
$$

where

$$
[X,Y]^\mu = \left(X^\nu \frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu} \right). \tag{2.45}
$$

Since f is arbitrary, it follows that

$$
[X,Y] = [X,Y]^\mu \left(\frac{\partial}{\partial x^\mu} \right). \tag{2.46}
$$

The RHS is a vector field hence $[X,Y]$ is a vector field whose components in a coordinate basis are given by (2.45). (Note that we *cannot* write equation (2.45) in abstract index notation because it is valid only in a *coordinate* basis.)

Example. Let $X = \partial/\partial x^1$ and $Y = x^1 \partial/\partial x^2 + \partial/\partial x^3$. The components of X are constant so $[X,Y]^\mu = \partial Y^\mu/\partial x^1 = \delta^\mu_2$ so $[X,Y] = \partial/\partial x^2$.

Exercise. Show that (i) $[X,Y] = -[Y,X]$; (ii) $[X, Y+Z] = [X,Y] + [X,Z]$; (iii) $[X, fY] = f[X,Y] + X(f)Y$; (iv) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ (the *Jacobi identity*). Here X, Y, Z are vector fields and f is a smooth function.

Remark. The components of $(\partial/\partial x^\mu)$ in the coordinate basis are either 1 or 0. It follows that

$$
\left[\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu} \right] = 0. \tag{2.47}
$$

Conversely, it can be shown that if $X_1, \ldots, X_m$ ($m \le n$) are vector fields that are linearly independent at every point, and whose commutators all vanish, then, in a neighbourhood of any point p, one can introduce a coordinate chart $(x^1, \ldots x^n)$ such that $X_i = \partial/\partial x^i$ ($i = 1, \ldots, m$) throughout this neighbourhood.

2.10 Integral curves

In fluid mechanics, the velocity of a fluid is described by a vector field $\mathbf{u}(\mathbf{x})$ in $\mathbb{R}^3$ (we are returning to Cartesian vector notation for a moment). Consider a particle suspended in the fluid with initial position $\mathbf{x}_0$. It moves with the fluid so its position $\mathbf{x}(\mathbf{t})$ satisfies

$$
\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}(t)), \qquad \mathbf{x}(0) = \mathbf{x}_0. \tag{2.48}
$$

The solution of this differential equation is called the *integral curve* of the vector field $\mathbf{u}$ through $\mathbf{x}_0$. The definition extends straightforwardly to a vector field on a general manifold:

Definition. Let X be a vector field on M and $p \in M$. An *integral curve of X through p* is a curve through p whose tangent at every point is X.

Let λ denote an integral curve of X with (wlog) $\lambda(0) = p$. In a coordinate chart, this definition reduces to the initial value problem

$$\frac{dx^\mu(t)}{dt} = X^\mu(x(t)), \qquad x^\mu(0) = x^\mu_p. \tag{2.49}$$

(Here we are using the abbreviation $x^\mu(t) = x^\mu(\lambda(t))$.) Standard ODE theory guarantees that there exists a unique solution to this problem. Hence there is a unique integral curve of X through any point p.

Example. In a chart $\phi = (x^1, \ldots, x^n)$, consider $X = \partial/\partial x^1 + x^1 \partial/\partial x^2$ and take p to be the point with coordinates $(0, \ldots, 0)$. Then $dx^1/dt = 1$, $dx^2/dt = x^1$. Solving the first equation and imposing the initial condition gives $x^1 = t$, then plugging into the second equation and solving gives $x^2 = t^2/2$. The other coords are trivial: $x^\mu = 0$ for $\mu > 2$, so the integral curve is $t \mapsto \phi^{-1}(t, t^2/2, 0, \ldots, 0)$.

Chapter 3

The metric tensor

3.1 Metrics

A metric captures the notion of *distance* on a manifold. We can motivate the required definition by considering the case of $\mathbb{R}^3$. Let $\mathbf{x}(t)$, $a < t < b$ be a curve in $\mathbb{R}^3$ (we're using Cartesian vector notation). Then the length of the curve is

$$\int_a^b dt \sqrt{\frac{d\mathbf{x}}{dt} \cdot \frac{d\mathbf{x}}{dt}}. \tag{3.1}$$

Inside the integral we see the norm of the tangent vector $d\mathbf{x}/dt$, in other words the scalar product of this vector with itself. Therefore to define a notion of distance on a general manifold, we shall start by introducing a scalar product between vectors.

A scalar product maps a pair of vectors to a number. In other words, at a point p, it is a map $g : T_p(M) \times T_p(M) \to \mathbb{R}$. A scalar product should be linear in each argument. Hence g is a $(0, 2)$ tensor at p. We call g a *metric tensor*. There are a couple of other properties that g should also satisfy:

Definition. A metric tensor at $p \in M$ is a $(0, 2)$ tensor g with the following properties:

1. It is symmetric: $g(X, Y) = g(Y, X)$ for all $X, Y \in T_p(M)$ (i.e. $g_{ab} = g_{ba}$)
2. It is non-degenerate: $g(X, Y) = 0$ for all $Y \in T_p(M)$ if, and only if, $X = 0$.

Remark. Sometimes we shall denote $g(X, Y)$ by $\langle X, Y \rangle$ or $X \cdot Y$.

Since the components of g form a symmetric matrix, one can introduce a basis that diagonalizes g. Non-degeneracy implies that none of the diagonal elements is zero. By rescaling the basis vectors, one can arrange that the diagonal elements are all ± 1. In this case, the basis is said to be *orthonormal*. There are many such bases but a standard algebraic theorem (Sylvester's law of inertia) states that the number of positive and negative elements is independent of the choice of orthonormal basis. The number of positive and negative elements is called the *signature* of the metric.

In differential geometry, one is usually interested in *Riemannian* metrics. These have signature $+++\ldots+$ (i.e. all diagonal elements $+1$ in an orthonormal basis), and hence g is positive definite. In GR, we are interested in *Lorentzian* metrics, i.e., those with signature $-++\ldots+$. This can be motivated by the equivalence principle as follows. Let spacetime be a 4d manifold M. Consider a local inertial frame (LIF) at p, with coordinates $\{x^\mu\}$. A pair of vectors X^a, Y^a at p have components X^μ, Y^μ w.r.t the coordinate basis of the LIF. The Einstein EP implies that special relativity holds in the LIF. In special relativity, we can define a scalar product $\eta_{\mu\nu}X^\mu Y^\nu$ where $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$. This is Lorentz invariant and hence gives the same result for all LIFs at p. So the EP predicts that we can define a (Lorentzian) scalar product at p, i.e., there exists a Lorentzian metric g at p which has components $\eta_{\mu\nu}$ in a LIF at p. We want g to be defined over the whole manifold, so we assume it to be a tensor field.

Definition. A *Riemannian (Lorentzian) manifold* is a pair (M, g) where M is a differentiable manifold and g is a Riemannian (Lorentzian) metric tensor field. A Lorentzian manifold is sometimes called a *spacetime*.

Remark. On a Riemannian manifold, we can now define the length of a curve in exactly the same way as above: let $\lambda : (a,b) \to M$ be a smooth curve with tangent vector X. Then the length of the curve is

$$\int_a^b dt \sqrt{g(X,X)|_{\lambda(t)}} \tag{3.2}$$

Exercise. Given a curve $\lambda(t)$ we can define a new curve simply by changing the parametrization: let $t = t(u)$ with $dt/du > 0$ and $u \in (c,d)$ with $t(c) = a$ and $t(d) = b$. Show that: (i) the new curve $\kappa(u) \equiv \lambda(t(u))$ has tangent vector $Y^a = (dt/du)X^a$; (ii) the length of these two curves is the same, i.e., our definition of length is independent of parametrization.

In a coordinate basis, we have (cf equation (2.34))

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu \tag{3.3}$$

Often we use the notation ds^2 instead of g and abbreviate this to

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{3.4}$$

This notation captures the intuitive idea of an infinitesimal distance ds being determined by infinitesimal coordinate separations dx^μ.

Examples.

1. In $\mathbb{R}^n = \{(x^1, \ldots, x^n)\}$, the *Euclidean metric* is

$$g = dx^1 \otimes dx^1 + \ldots + dx^n \otimes dx^n \tag{3.5}$$

   $(\mathbb{R}^n, g)$ is callled *Euclidean space.* A coordinate chart which covers all of $\mathbb{R}^4$ and in which the components of the metric are $\text{diag}(1, 1, \ldots, 1)$ is called *Cartesian.*

2. In $\mathbb{R}^4 = \{(x^0, x^1, x^2, x^3)\}$, the *Minkowski metric* is

$$\eta = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \tag{3.6}$$

   $(\mathbb{R}^4, \eta)$ is called *Minkowski spacetime.* A coordinate chart which covers all of $\mathbb{R}^4$ and in which the components of the metric are $\eta_{\mu\nu} \equiv \text{diag}(-1, 1, 1, 1)$ everywhere is called an *inertial frame.*

3. On S^2, let (θ, ϕ) denote the spherical polar coordinate chart discussed earlier. The (unit) *round metric* on S^2 is

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2, \tag{3.7}$$

   i.e. in the chart (θ, ϕ), we have $g_{\mu\nu} = \text{diag}(1, \sin^2\theta)$. Note this is positive definite for $\theta \in (0, \pi)$, i.e., on all of this chart. However, this chart does not cover the whole manifold so the above equation does not determine g everywhere. We can give a precise definition by adding that, in the chart (θ', ϕ') discussed earlier, $g = d\theta'^2 + \sin^2\theta' d\phi'^2$. One can check that this does indeed define a smooth tensor field. (This metric is the one induced from the embedding of S^2 into 3d Euclidean space: it is the "pull-back" of the metric on Euclidean space - see later for the definition of pull-back.)

Definition. Since g_{ab} is non-degenerate, it must be invertible. The *inverse metric* is a symmetric $(2, 0)$ tensor field denoted g^{ab} and obeys

$$g^{ab} g_{bc} = \delta^a_c \tag{3.8}$$

Example. For the metric on S^2 defined above, in the chart (θ, ϕ) we have $g^{\mu\nu} = \text{diag}(1, 1/\sin^2\theta)$.

Definition. A metric determines a natural isomorphism between vectors and covectors. Given a vector X^a we can define a covector $X_a = g_{ab} X^b$. Given a covector η_a we can define a vector $\eta^a = g^{ab} \eta_b$. These maps are clearly inverses of each other.

Remark. This isomorphism is the reason why covectors are not more familiar: we are used to working in Euclidean space using Cartesian coordinates, for which $g_{\mu\nu}$ and $g^{\mu\nu}$ are both the identity matrix, so the isomorphism appears trivial.

Definition. For a general tensor, abstract indices can be "lowered" by contracting with g_{ab} and "raised" by contracting with g^{ab}. Raising and lowering preserve the ordering of indices. The resulting tensor will be denoted by the same letter as the original tensor.

Example. Let T be a $(3,2)$ tensor. Then $T^a{}_b{}^{cde} = g_{bf} g^{dh} g^{ej} T^{afc}{}_{hj}$.

3.2 Lorentzian signature

Remark. On a Lorentzian manifold, we take basis indices $\mu, \nu, \ldots$ to run from 0 to $n-1$.

At any point p of a Lorentzian manifold, we can choose an orthonormal basis $\{e_\mu\}$ so that $g(e_\mu, e_\nu) = \eta_{\mu\nu} \equiv \text{diag}(-1, 1, \ldots, 1)$. Such a basis is far from unique. If $e'_\mu = (A^{-1})^\nu{}_\mu e_\nu$ is any other such basis then we have

$$\eta_{\mu\nu} = g(e'_\mu, e'_\nu) = (A^{-1})^\rho{}_\mu (A^{-1})^\sigma{}_\nu g(e_\rho, e_\sigma) = (A^{-1})^\rho{}_\mu (A^{-1})^\sigma{}_\nu \eta_{\rho\sigma}. \tag{3.9}$$

Hence

$$\eta_{\mu\nu} A^\mu{}_\rho A^\nu{}_\sigma = \eta_{\rho\sigma}. \tag{3.10}$$

These are the defining equations of a *Lorentz transformation* in special relativity. Hence different orthonormal bases at p are related by Lorentz transformations. We saw earlier that the components of a vector at p transform as $X'^\mu = A^\mu{}_\nu X^\nu$. We are starting to recover the structure of special relativity *locally*, as required by the Equivalence Principle.

Definition. On a Lorentzian manifold (M, g), a non-zero vector $X \in T_p(M)$ is *timelike* if $g(X,X) < 0$, *null* (or *lightlike*) if $g(X,X) = 0$, and *spacelike* if $g(X,X) > 0$.

Remark. In an orthonormal basis at p, the metric has components $\eta_{\mu\nu}$ so the tangent space at p has exactly the same structure as Minkowski spacetime, i.e., null vectors at p define a light cone that separates timelike vectors at p from spacelike vectors at p (see Fig. 3.1).

Exercise. Let X^a, Y^b be non-zero vectors at p that are orthogonal, i.e., $g_{ab} X^a Y^b = 0$. Show that (i) if X^a is timelike then Y^a is spacelike; (ii) if X^a is null then Y^a is spacelike or null; (iii) if X^a is spacelike then Y^a can be spacelike, timelike, or null. (*Hint.* Choose an orthonormal basis to make the components of X^a as simple as possible.)

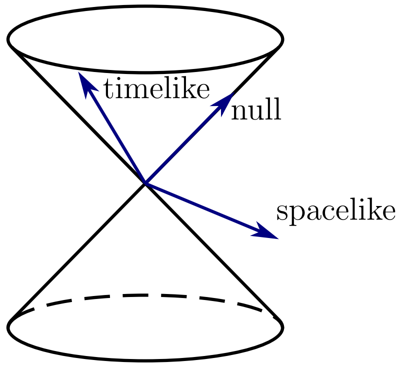


Figure 3.1: Light cone structure of $T_p(M)$

Definition. On a Riemannian manifold, the *norm* of a vector X is $|X| = \sqrt{g(X,X)}$ and the *angle* between two non-zero vectors X and Y (at the same point) is θ where $\cos\theta = g(X,Y)/(|X|\,|Y|)$.

Definition. A curve in a Lorentzian manifold is said to be *timelike* if its tangent vector is everywhere timelike. Null and spacelike curves are defined similarly. (Most curves do not satisfy any of these definitions because e.g. the tangent vector can change from timelike to null to spacelike along a curve.)

Remark. The length of a spacelike curve can be defined in exactly the same way as on a Riemannian manifold (equation (3.2)). What about a timelike curve?

Definition. let $\lambda(u)$ be a timelike curve with $\lambda(0) = p$. Let X^a be the tangent to the curve. The *proper time* τ from p along the curve is defined by

$$\frac{d\tau}{du} = \sqrt{-\left(g_{ab}X^a X^b\right)_{\lambda(u)}}, \qquad \tau(0) = 0. \tag{3.11}$$

Remark. In a coordinate chart, $X^\mu = dx^\mu/du$ so this definition can be rewritten in the form

$$d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu, \tag{3.12}$$

with the understanding that this is to be evaluated along the curve. Integrating the above equation along the curve gives the proper time from p to some other point $q = \lambda(u_q)$ as

$$\tau = \int_0^{u_q} du \sqrt{-\left(g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du}\right)_{\lambda(u)}} \tag{3.13}$$

Definition. If proper time τ is used to parametrize a timelike curve then the tangent to the curve is called the *4-velocity* of the curve. In a coordinate basis, it has components $u^\mu = dx^\mu/d\tau$.

Remark. (3.12) implies that 4-velocity is a unit timelike vector:

$$g_{ab}u^a u^b = -1. \tag{3.14}$$

3.3 Curves of extremal proper time

Consider the following question. Let p and q be points connected by a timelike curve. A small deformation of a timelike curve remains timelike hence there exist infinitely many timelike curves connecting p and q. The proper time between p and q will be different for different curves. Which curve extremizes the proper time between p and q?

This is a standard Euler-Lagrange problem. Consider timelike curves from p to q with parameter u such that $\lambda(0) = p$, $\lambda(1) = q$. Let's use a dot to denote a derivative with respect to u. The proper time between p and q along such a curve is given by the functional

$$\tau[\lambda] = \int_0^1 du\, G\left(x(u), \dot{x}(u)\right) \tag{3.15}$$

where

$$G\left(x(u), \dot{x}(u)\right) \equiv \sqrt{-g_{\mu\nu}(x(u))\dot{x}^\mu(u)\dot{x}^\nu(u)} \tag{3.16}$$

and we are writing $x^\mu(u)$ as a shorthand for $x^\mu(\lambda(u))$.

The curve that extremizes the proper time, must satisfy the Euler-Lagrange equation

$$\frac{d}{du}\left(\frac{\partial G}{\partial \dot{x}^\mu}\right) - \frac{\partial G}{\partial x^\mu} = 0 \tag{3.17}$$

Working out the various terms, we have (using the symmetry of the metric)

$$\frac{\partial G}{\partial \dot{x}^\mu} = -\frac{1}{2G} 2 g_{\mu\nu}\dot{x}^\nu = -\frac{1}{G} g_{\mu\nu}\dot{x}^\nu \tag{3.18}$$

$$\frac{\partial G}{\partial x^\mu} = -\frac{1}{2G} g_{\nu\rho,\mu}\,\dot{x}^\nu \dot{x}^\rho \tag{3.19}$$

where we have relabelled some dummy indices, and introduced the important notation of a comma to denote partial differentiation:

$$g_{\nu\rho,\mu} \equiv \frac{\partial}{\partial x^\mu} g_{\nu\rho} \tag{3.20}$$

We will be using this notation a lot henceforth.

So far, our parameter u has been arbitrary subject to the conditions $u(0) = p$ and $u(1) = q$. At this stage, it is convenient to use a more physical parameter, namely τ, the proper time along the curve. (Note that we could not have used τ from the outset since the value of τ at q is different for different curves, which would make the range of integration different for different curves.) The paramers are related by

$$\left(\frac{d\tau}{du}\right)^2 = -g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = G^2 \tag{3.21}$$

and hence $d\tau/du = G$. So in our equations above, we can replace d/du with $Gd/d\tau$, so the Euler-Lagrange equation becomes (after cancelling a factor of $-G$)

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) - \frac{1}{2}g_{\nu\rho,\mu}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \tag{3.22}$$

Hence

$$g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} + g_{\mu\nu,\rho}\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau} - \frac{1}{2}g_{\nu\rho,\mu}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \tag{3.23}$$

In the second term, we can replace $g_{\mu\nu,\rho}$ with $g_{\mu(\nu,\rho)}$ because it is contracted with an object symmetrical on ν and ρ. Finally, contracting the whole expression with the inverse metric and relabelling indices gives

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \tag{3.24}$$

where $\Gamma^\mu_{\nu\rho}$ are known as the *Christoffel symbols*, and are defined by

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}\right). \tag{3.25}$$

Remarks. 1. $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$. 2. The Christoffel symbols are *not* tensor components. Neither the first term nor the second term in (3.24) are components of a vector but the sum of these two terms does give vector components. More about this soon. 3. Equation 3.24 is called the *geodesic equation*. Geodesics will be defined below.

Example. In Minkowski spacetime, the components of the metric in an inertial frame are constant so $\Gamma^\mu_{\nu\rho} = 0$. Hence the above equation reduces to $d^2x^\mu/d\tau^2 = 0$. This is the equation of motion of a free particle! Hence, in Minkowski spacetime, the free particle trajectory between two (timelike separated) points p and q extremizes the proper time between p and q.

This motivates the following postulate of General Relativity:

Postulate. Massive test bodies follow curves of extremal proper time, i.e., solutions of equation (3.24).

Remarks. 1. Massless particles obey a very similar equation which we shall discuss shortly. 2. In Minkowski spacetime, curves of extremal proper time *maximize* the proper time between two points. In a curved spacetime, this is true only *locally*, i.e., for any point p there exists a neighbourhood of p within which it is true.

Exercises

1. Show that (3.24) can be obtained more directly as the Euler-Lagrange equation for the Lagrangian

$$L = -g_{\mu\nu}(x(\tau))\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{3.26}$$

   This is usually the easiest way to derive (3.24) or to calculate the Christoffel symbols.

2. Note that L has no explicit τ dependence, i.e., $\partial L/\partial\tau = 0$. Show that this implies that the following quantity is conserved along curves of extremal proper time (i.e. that is is annihilated by $d/d\tau$):

$$L - \frac{\partial L}{\partial(dx^\mu/d\tau)}\frac{dx^\mu}{d\tau} = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{3.27}$$

   This is a check on the consistency of (3.24) because the definition of τ as proper time implies that the RHS must be -1.

Example. The Schwarzschild metric in Schwarzschild coordinates (t, r, θ, ϕ) is

$$ds^2 = -f dt^2 + f^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2, \qquad f = 1 - \frac{2M}{r} \tag{3.28}$$

where M is a constant. We have

$$L = f\left(\frac{dt}{d\tau}\right)^2 - f^{-1}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\theta}{d\tau}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 \tag{3.29}$$

so the EL equation for $t(\tau)$ is

$$\frac{d}{d\tau}\left(2f\frac{dt}{d\tau}\right) = 0 \qquad \Rightarrow \qquad \frac{d^2t}{d\tau^2} + f^{-1}f'\frac{dt}{d\tau}\frac{dr}{d\tau} = 0 \tag{3.30}$$

From this we can read off

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{f'}{2f}, \qquad \Gamma^0_{\mu\nu} = 0 \text{ otherwise} \tag{3.31}$$

The other Christoffel symbols are obtained in a similar way from the remaining EL equations (examples sheet 1).

Chapter 4

Covariant derivative

4.1 Introduction

To formulate physical laws, we need to be able to differentiate tensor fields. For scalar fields, partial differentiation is fine: $f_{,\mu} \equiv \partial f/\partial x^\mu$ are the components of the covector field $(df)_a$. However, for tensor fields, partial differentiation is no good because the partial derivative of a tensor field does not give another tensor field:

Exercise. Let V^a be a vector field. In any coordinate chart, let $T^\mu{}_\nu = V^\mu{}_{,\nu} \equiv \partial V^\mu/\partial x^\nu$. Show that $T^\mu{}_\nu$ do *not* transform as tensor components under a change of chart.

The problem is that differentiation involves comparing a tensor at two infinitesimally nearby points of the manifold. But we have seen that this does not make sense: tensors at different points belong to different spaces. The mathematical structure that overcomes this difficulty is called a *covariant derivative* or *connection*.

Definition. A *covariant derivative* ∇ on a manifold M is a map sending every pair of smooth vector fields X, Y to a smooth vector field $\nabla_X Y$, with the following properties (where X, Y, Z are vector fields and f, g are functions)

$$\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z, \tag{4.1}$$

$$\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z, \tag{4.2}$$

$$\nabla_X (fY) = f\nabla_X Y + (\nabla_X f)Y, \qquad \text{(Leibniz rule)}, \tag{4.3}$$

where the action of ∇ on functions is defined by

$$\nabla_X f = X(f). \tag{4.4}$$

Remark. (4.1) implies that, at any point, the map $\nabla Y : X \mapsto \nabla_X Y$ is a linear map from $T_p(M)$ to itself. Hence it defines a $(1,1)$ tensor (see examples sheet 1). More precisely, if $\eta \in T_p^*(M)$ and $X \in T_p(M)$ then we define $(\nabla Y)(\eta, X) \equiv \eta(\nabla_X Y)$.

Definition. let Y be a vector field. The *covariant derivative of* Y is the $(1,1)$ tensor field ∇Y. In abstract index notation we usually write $(\nabla Y)^a{}_b$ as $\nabla_b Y^a$ or $Y^a{}_{;b}$

Remarks.

1. Similarly we define $\nabla f : X \mapsto \nabla_X f = X(f)$. Hence $\nabla f = df$. We can write this as either $\nabla_a f$ or $f_{;a}$ or $\partial_a f$ or $f_{,a}$ (i.e. the covariant derivative reduces to the partial derivative when acting on a function).

2. Does the map $\nabla : X, Y \mapsto \nabla_X Y$ define a $(1,2)$ tensor field? No - equation (4.3) shows that this map is not linear in Y.

Example. Pick a coordinate chart on M. Let ∇ be the partial derivative in this chart. This satisfies all of the above conditions. This is not a very interesting example of a covariant derivative because it depends on choosing a particular chart: if we use a different chart then this covariant derivative will *not* be the partial derivative in the new chart.

Definition. In a basis $\{e_\mu\}$ the *connection components* $\Gamma^\mu_{\nu\rho}$ are defined by

$$\nabla_\rho e_\nu \equiv \nabla_{e_\rho} e_\nu = \Gamma^\mu_{\nu\rho} e_\mu \tag{4.5}$$

Example. The Christoffel symbols are the coordinate basis components of a certain connection, the *Levi-Civita connection*, which is defined on any manifold with a metric. More about this soon.

Write $X = X^\mu e_\mu$ and $Y = Y^\mu e_\mu$. Now

$$\begin{aligned}
\nabla_X Y &= \nabla_X (Y^\mu e_\mu) = X(Y^\mu) e_\mu + Y^\mu \nabla_X e_\mu \qquad \text{(Leibniz)} \\
&= X^\nu e_\nu(Y^\mu) e_\mu + Y^\mu \nabla_{X^\nu e_\nu} e_\mu \\
&= X^\nu e_\nu(Y^\mu) e_\mu + Y^\mu X^\nu \nabla_\nu e_\mu \qquad \text{by (4.1)} \\
&= X^\nu e_\nu(Y^\mu) e_\mu + Y^\mu X^\nu \Gamma^\rho_{\mu\nu} e_\rho \\
&= X^\nu \left(e_\nu(Y^\mu) + \Gamma^\mu_{\rho\nu} Y^\rho \right) e_\mu
\end{aligned} \tag{4.6}$$

and hence

$$(\nabla_X Y)^\mu = X^\nu e_\nu(Y^\mu) + \Gamma^\mu_{\rho\nu} Y^\rho X^\nu \tag{4.7}$$

so

$$Y^\mu{}_{;\nu} = e_\nu(Y^\mu) + \Gamma^\mu_{\rho\nu} Y^\rho \tag{4.8}$$

In a coordinate basis, this reduces to

$$Y^\mu{}_{;\nu} = Y^\mu{}_{,\nu} + \Gamma^\mu_{\rho\nu} Y^\rho \tag{4.9}$$

The connection components $\Gamma^\mu_{\nu\rho}$ are not tensor components:

Exercise (examples sheet 2). Consider a change of basis $e'_\mu = (A^{-1})^\nu{}_\mu e_\nu$. Show that

$$\Gamma'^\mu_{\nu\rho} = A^\mu{}_\tau (A^{-1})^\lambda{}_\nu (A^{-1})^\sigma{}_\rho \Gamma^\tau_{\lambda\sigma} + A^\mu{}_\tau (A^{-1})^\sigma{}_\rho e_\sigma((A^{-1})^\tau{}_\nu) \tag{4.10}$$

The presence of the second term demonstrates that $\Gamma^\mu_{\nu\rho}$ are not tensor components. Hence neither term in the RHS of equation (4.9) transforms as a tensor. However, the *sum* of these two terms does transform as a tensor.

Exercise. Let ∇ and $\tilde{\nabla}$ be two different connections on M. Show that $\nabla - \tilde{\nabla}$ is a $(1,2)$ tensor field. You can do this either from the definition of a connection, or from the transformation law for the connection components.

The action of ∇ is extended to general tensor fields by the Leibniz property. If T is a tensor field of type (r,s) then ∇T is a tensor field of type $(r, s+1)$. For example, if η is a covector field then, for any vector fields X and Y, we define

$$(\nabla_X \eta)(Y) \equiv \nabla_X(\eta(Y)) - \eta(\nabla_X Y). \tag{4.11}$$

It is not obvious that this defines a $(0,2)$ tensor but we can see this as follows:

$$\begin{aligned}(\nabla_X \eta)(Y) &= \nabla_X(\eta_\mu Y^\mu) - \eta_\mu (\nabla_X Y)^\mu \\ &= X(\eta_\mu) Y^\mu + \eta_\mu X(Y^\mu) - \eta_\mu \left(X^\nu e_\nu(Y^\mu) + \Gamma^\mu_{\rho\nu} Y^\rho X^\nu \right),\end{aligned} \tag{4.12}$$

where we used (4.7). Now, the second and third terms cancel ($X = X^\nu e_\nu$) and hence (renaming dummy indices in the final term)

$$(\nabla_X \eta)(Y) = \left(X(\eta_\mu) - \Gamma^\rho_{\mu\nu} \eta_\rho X^\nu \right) Y^\mu, \tag{4.13}$$

which is linear in Y^μ so $\nabla_X \eta$ is a covector field with components

$$\begin{aligned}(\nabla_X \eta)_\mu &= X(\eta_\mu) - \Gamma^\rho_{\mu\nu} \eta_\rho X^\nu \\ &= X^\nu \left(e_\nu(\eta_\mu) - \Gamma^\rho_{\mu\nu} \eta_\rho \right)\end{aligned} \tag{4.14}$$

This is linear in X^ν and hence $\nabla \eta$ is a $(0,2)$ tensor field with components

$$\eta_{\mu;\nu} = e_\nu(\eta_\mu) - \Gamma^\rho_{\mu\nu} \eta_\rho \tag{4.15}$$

In a coordinate basis, this is

$$\eta_{\mu;\nu} = \eta_{\mu,\nu} - \Gamma^\rho_{\mu\nu} \eta_\rho \tag{4.16}$$

Now the Leibniz rule can be used to obtain the formula for the coordinate basis components of ∇T where T is a (r,s) tensor:

$$\begin{aligned} T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s;\rho} &= T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s,\rho} + \Gamma^{\mu_1}_{\sigma\rho} T^{\sigma\mu_2\ldots\mu_r}{}_{\nu_1\ldots\nu_s} + \ldots + \Gamma^{\mu_r}_{\sigma\rho} T^{\mu_1\ldots\mu_{r-1}\sigma}{}_{\nu_1\ldots\nu_s} \\ &- \Gamma^{\sigma}_{\nu_1\rho} T^{\mu_1\ldots\mu_r}{}_{\sigma\nu_2\ldots\nu_s} - \ldots - \Gamma^{\sigma}_{\nu_s\rho} T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_{s-1}\sigma} \end{aligned} \tag{4.17}$$

Exercise. Prove this result for a $(1,1)$ tensor.

Remark. We are using a comma and semi-colon to denote partial, and covariant, derivatives respectively. If more than one index appears after a comma or semi-colon then the derivative is to be taken with respect to all indices. The index nearest to comma/semi-colon is the *first* derivative to be taken. For example, $f_{,\mu\nu} = f_{,\mu,\nu} \equiv \partial_\nu \partial_\mu f$, and $X^a{}_{;bc} = \nabla_c \nabla_b X^a$ (we cannot use abstract indices for the first example since it is not a tensor). The second partial derivatives of a function commute: $f_{,\mu\nu} = f_{,\nu\mu}$ but for a covariant derivative this is not true in general. Set $\eta = df$ in (4.16) to get, in a coordinate basis,

$$f_{;\mu\nu} = f_{,\mu\nu} - \Gamma^\rho_{\mu\nu} f_{,\rho} \tag{4.18}$$

Antisymmetrizing gives

$$f_{;[\mu\nu]} = -\Gamma^\rho_{[\mu\nu]} f_{,\rho} \qquad \text{(coordinate basis)} \tag{4.19}$$

Definition. A connection ∇ is *torsion-free* if $\nabla_a \nabla_b f = \nabla_b \nabla_a f$ for any function f. From (4.19), this is equivalent to

$$\Gamma^\rho_{[\mu\nu]} = 0 \qquad \text{(coordinate basis)} \tag{4.20}$$

Lemma. For a torsion-free connection, if X and Y are vector fields then

$$\nabla_X Y - \nabla_Y X = [X,Y] \tag{4.21}$$

Proof. Use a coordinate basis:

$$\begin{aligned} X^\nu Y^\mu{}_{;\nu} - Y^\nu X^\mu{}_{;\nu} &= X^\nu Y^\mu{}_{,\nu} + \Gamma^\mu_{\rho\nu} X^\nu Y^\rho - Y^\nu X^\mu{}_{,\nu} - \Gamma^\mu_{\rho\nu} Y^\nu X^\rho \\ &= [X,Y]^\mu + 2\Gamma^\mu_{[\rho\nu]} X^\nu Y^\rho \\ &= [X,Y]^\mu \end{aligned} \tag{4.22}$$

Hence the equation is true in a coordinate basis and therefore (as it is a tensor equation) it is true in any basis.

Remark. Even with zero torsion, the second covariant derivatives of a *tensor* field do *not* commute. More soon.

4.2 The Levi-Civita connection

On a manifold with a metric, the metric singles out a preferred connection:

Theorem. Let M be a manifold with a metric g. There exists a unique torsion-free connection ∇ such that the metric is covariantly constant: $\nabla g = 0$ (i.e. $g_{ab;c} = 0$). This is called the *Levi-Civita (or metric) connection*.

Proof. Let X, Y, Z be vector fields then

$$X(g(Y,Z)) = \nabla_X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \tag{4.23}$$

where we used the Leibniz rule and $\nabla_X g = 0$ in the second equality. Permuting X, Y, Z leads to two similar identities:

$$Y(g(Z,X)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X), \tag{4.24}$$

$$Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y), \tag{4.25}$$

Add the first two of these equations and subtract the third to get (using the symmetry of the metric)

$$\begin{aligned} X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) &= g(\nabla_X Y + \nabla_Y X, Z) \\ &- g(\nabla_Z X - \nabla_X Z, Y) \\ &+ g(\nabla_Y Z - \nabla_Z Y, X) \end{aligned} \tag{4.26}$$

The torsion-free condition implies

$$\nabla_X Y - \nabla_Y X = [X,Y] \tag{4.27}$$

Using this and the same identity with X, Y, Z permuted gives

$$\begin{aligned} X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) &= 2g(\nabla_X Y, Z) - g([X,Y],Z) \\ &- g([Z,X],Y) + g([Y,Z],X) \end{aligned} \tag{4.28}$$

Hence

$$\begin{aligned} g(\nabla_X Y, Z) &= \frac{1}{2}[X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &+ g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X)] \end{aligned} \tag{4.29}$$

This determines $\nabla_X Y$ uniquely because the metric is non-degenerate. It remains to check that it satisfies the properties of a connection. For example:

$$g(\nabla_{fX} Y, Z) = \frac{1}{2}[fX(g(Y,Z)) + Y(fg(Z,X)) - Z(fg(X,Y))$$

$$
\begin{aligned}
&+ \quad g([fX,Y],Z) + g([Z,fX],Y) - fg([Y,Z],X)] \\
&= \quad \frac{1}{2}\left[fX(g(Y,Z)) + fY(g(Z,X)) + Y(f)g(Z,X)\right. \\
&- \quad fZ(g(X,Y)) - Z(f)g(X,Y) + fg([X,Y],Z) - Y(f)g(X,Z) \\
&+ \quad \left. fg([Z,X],Y) + Z(f)g(X,Y) - fg([Y,Z],X)\right] \\
&= \quad \frac{f}{2}\left[X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))\right. \\
&+ \quad \left. g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X)\right] \\
&= \quad fg(\nabla_X Y, Z) = g(f\nabla_X Y, Z)
\end{aligned}
\tag{4.30}
$$

and hence $g(\nabla_{fX}Y - f\nabla_X Y, Z) = 0$ for any vector field Z so, by the non-degeneracy of the metric, $\nabla_{fX}Y = f\nabla_X Y$.

Exercise. Show that $\nabla_X Y$ as defined by (4.29) satisfies the other properties required of a connection.

Remark. In differential geometry, this theorem is called the *fundamental theorem of Riemannian geometry* (although it applies for a metric of any signature).

Let's determine the components of the Levi-Civita connection in a coordinate basis (for which $[e_\mu, e_\nu] = 0$):

$$
g(\nabla_\rho e_\nu, e_\sigma) = \frac{1}{2}\left[e_\rho(g_{\nu\sigma}) + e_\nu(g_{\sigma\rho}) - e_\sigma(g_{\rho\nu})\right], \tag{4.31}
$$

that is

$$
g(\Gamma^\tau_{\nu\rho} e_\tau, e_\sigma) = \frac{1}{2}\left(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}\right) \tag{4.32}
$$

The LHS is just $\Gamma^\tau_{\nu\rho} g_{\tau\sigma}$. Hence if we multiply the whole equation by the inverse metric $g^{\mu\sigma}$ we obtain

$$
\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}\left(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}\right) \tag{4.33}
$$

This is the same equation as we obtained earlier; we have now shown that the Christoffel symbols are the components of the Levi-Civita connection.

Remark. In GR, we take the connection to be the Levi-Civita connection. This is not as restrictive as it sounds: we saw above that the difference between two connections is a tensor field. Hence we can write any connection (even one with torsion) in terms of the Levi-Civita connection and a $(1,2)$ tensor field. In GR we could regard the latter as a particular kind of "matter" field, rather than as part of the geometry of spacetime.

4.3 Geodesics

Previously we considered curves that extremize the proper time between two points of a spacetime, and showed that this gives the equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x(\tau))\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0, \tag{4.34}$$

where τ is the proper time along the curve. The tangent vector to the curve has components $X^\mu = dx^\mu/d\tau$. This is defined only along the curve. However, we can extend X^μ (in an arbitrary way) to a neighbourhood of the curve, so that X^μ becomes a vector field, and the curve is an integral curve of this vector field. The chain rule gives

$$\frac{d^2x^\mu}{d\tau^2} = \frac{dX^\mu(x(\tau))}{d\tau} = \frac{dx^\nu}{d\tau}\frac{\partial X^\mu}{\partial x^\nu} = X^\nu X^\mu{}_{,\nu}. \tag{4.35}$$

Note that the LHS is independent of how we extend X^μ hence so must be the RHS. We can now write (4.34) as

$$X^\nu \left(X^\mu{}_{,\nu} + \Gamma^\mu_{\nu\rho} X^\rho \right) = 0 \tag{4.36}$$

which is the same as

$$X^\nu X^\mu{}_{;\nu} = 0, \qquad \text{or} \qquad \nabla_X X = 0. \tag{4.37}$$

where we are using the Levi-Civita connection. We now extend this to an arbitrary connection:

Definition. Let M be a manifold with a connection ∇. An *affinely parameterized geodesic* is an integral curve of a vector field X satisfying $\nabla_X X = 0$.

Remarks.

1. What do we mean by "affinely parameterized"? Consider a curve with parameter t whose tangent X satisfies the above definition. Let u be some other parameter for the curve, so $t = t(u)$ and $dt/du > 0$. Then the tangent vector becomes $Y = hX$ where $h = dt/du$. Hence

   $$\nabla_Y Y = \nabla_{hX}(hX) = h\nabla_X(hX) = h^2\nabla_X X + X(h)hX = fY, \tag{4.38}$$

   where $f = X(h) = dh/dt$. Hence $\nabla_Y Y = fY$ describes the same geodesic. In this case, the geodesic is *not* affinely parameterized.

   It always is possible to find an affine parameter so there is no loss of generality in restricting to affinely parameterized geodesics. Note that the new parameter also is affine iff $X(h) = 0$, i.e., h is constant. Then $u = at+b$ where a and b are constants with $a > 0$ ($a = h^{-1}$). Hence there is a 2-parameter family of affine parameters for any geodesic.

2. Reversing the above steps shows that, in a coordinate chart, for any connection, the geodesic equation can be written as (4.34) with τ an arbitrary affine parameter.

3. In GR, curves of extremal proper time are timelike geodesics (with ∇ the Levi-Civita connection). But one can also consider geodesics which are not timelike. These satisfy (4.34) with τ an affine parameter. The easiest way to obtain this equation is to use the Lagrangian (3.26).

Theorem. Let M be a manifold with a connection ∇. Let $p \in M$ and $X_p \in T_p(M)$. Then there exists a unique affinely parameterized geodesic through p with tangent vector X_p at p.

Proof. Choose a coordinate chart x^μ in a neighbourhood of p. Consider a curve parameterized by τ. It has tangent vector with components $X^\mu = dx^\mu/d\tau$. The geodesic equation is (4.34). We want the curve to satisfy the initial conditions

$$x^\mu(0) = x^\mu_p, \qquad \left(\frac{dx^\mu}{d\tau}\right)_{\tau=0} = X^\mu_p. \tag{4.39}$$

This is a coupled system of n ordinary differential equations for the n functions $x^\mu(t)$. Existence and uniqueness is guaranteed by the standard theory of ordinary differential equations.

Exercise. Let X be tangent to an affinely parameterized geodesic of the Levi-Civita connection. Show that $\nabla_X(g(X,X)) = 0$ and hence $g(X,X)$ is constant along the geodesic. Therefore the tangent vector cannot change e.g. from timelike to null along the geodesic: a geodesic is either timelike, spacelike or null.

Postulate. In GR, free particles move on geodesics (of the Levi-Civita connection). These are timelike for massive particles, and null for massless particles (e.g. photons).

Remark. In the timelike case we can use proper time as an affine parameter. This imposes the additional restriction $g(X,X) = -1$. If τ and τ' both are proper times along a geodesic then $\tau' = \tau + b$ (i.e. $a = 1$ above). In other words, clocks measuring proper time differ only by their choice of zero. In particular, they measure equal time intervals. Similarly in the spacelike case (or on a Riemannian manifold), we use arc length s as affine parameter, which gives $g(X,X) = 1$ and $s' = s + b$. In the null case, there is no analogue of proper time or arc length and so there is a 2-parameter ambiguity in affine parameterization.

4.4 Normal coordinates

Definition. Let M be a manifold with a connection ∇. Let $p \in M$. The *exponential map* from $T_p(M)$ to M is defined as the map which sends X_p to the point unit affine parameter distance along the geodesic through p with tangent X_p at p.

Remark. It can be shown that this map is one-to-one and onto *locally*, i.e., for X_p in a neighbourhood of the origin in $T_p(M)$.

Exercise. Let $0 \leq t \leq 1$. Show that the exponential map sends tX_p to the point affine parameter distance t along the geodesic through p with tangent X_p at p.

Definition. Let $\{e_\mu\}$ be a basis for $T_p(M)$. *Normal coordinates at p* are defined in a neighbourhood of p as follows. Pick q near p. Then the coordinates of q are X^μ where X^a is the element of $T_p(M)$ that maps to q under the exponential map.

Lemma. $\Gamma^\mu_{(\nu\rho)}(p) = 0$ in normal coordinates at p. For a torsion-free connection, $\Gamma^\mu_{\nu\rho}(p) = 0$ in normal coordinates at p.

Proof. From the above exercise, it follows that affinely parameterized geodesics through p are given in normal coordinates by $X^\mu(t) = tX^\mu_p$. Hence the geodesic equation reduces to

$$\Gamma^\mu_{\nu\rho}(X(t))X^\nu_p X^\rho_p = 0. \tag{4.40}$$

Evaluating at $t = 0$ gives that $\Gamma^\mu_{\nu\rho}(p)X^\nu_p X^\rho_p = 0$. But X_p is arbitrary, so the first result follows. The second result follows using the fact that torsion-free implies $\Gamma^\mu_{[\nu\rho]} = 0$ in a coordinate chart.

Remark. The connection components away from p *will not* vanish in general.

Lemma. On a manifold with a metric, if the Levi-Civita connection is used to define normal coordinates at p then $g_{\mu\nu,\rho} = 0$ at p.

Proof. Apply the previous lemma. We then have, at p,

$$0 = 2g_{\mu\sigma}\Gamma^\sigma_{\nu\rho} = g_{\mu\nu,\rho} + g_{\mu\rho,\nu} - g_{\nu\rho,\mu} \tag{4.41}$$

Now symmetrize on $\mu\nu$: the final two terms cancel and the result follows.

Remark. Again, we emphasize, this is valid *only at the point p*. At any point, we can introduce normal coordinates to make the first partial derivatives of the metric vanish at that point. They will not vanish away for that point.

Lemma. On a manifold with metric one can choose normal coordinates at p so that $g_{\mu\nu,\rho}(p) = 0$ and also $g_{\mu\nu}(p) = \eta_{\mu\nu}$ (Lorentzian case) or $g_{\mu\nu}(p) = \delta_{\mu\nu}$ (Riemannian case).

Proof. We've already shown $g_{\mu\nu,\rho}(p) = 0$. Consider $\partial/\partial X^1$. The integral curve through p of this vector field is $X^\mu(t) = (t, 0, 0, \ldots, 0)$ (since $X^\mu = 0$ at p). But, from the above, this is the same as the geodesic through p with tangent vector e_1 at p. It follows that $\partial/\partial X^1 = e_1$ at p (since both vectors are tangent to the curve at p). Similarly $\partial/\partial X^\mu = e_\mu$ at p. But the choice of basis $\{e_\mu\}$ was arbitrary. So we are free to choose $\{e_\mu\}$ to be an orthonormal basis. $\partial/\partial X^\mu$ then defines an orthonormal basis at p too.

In summary, on a Lorentzian (Riemannian) manifold, we can choose coordinates in the neighbourhood of any point p so that the components of the metric at p are the same as those of the Minkowski metric in inertial coordinates (Euclidean metric in Cartesian coordinates), and the first partial derivatives of the metric vanish at p.

Definition. In a Lorentzian manifold a *local inertial frame at* p is a set of normal coordinates at p with the above properties.

Thus the assumption that spacetime is a Lorentzian manifold leads to a precise mathematical definition of a local inertial frame.

Chapter 5

Physical laws in curved spacetime

5.1 Minimal coupling, equivalence principle

Physical laws in curved spacetime should exhibit *general covariance*: they should be independent of any choice of basis or coordinate chart. In special relativity, we restrict attention to coordinate systems corresponding to inertial frames. The laws of physics should exhibit *special covariance*, i.e, take the same form in any inertial frame (this is the principle of relativity). The following procedure can be used to convert such laws of physics into generally covariant laws:

1. Replace the Minkowski metric by a curved spacetime metric.
2. Replace partial derivatives with covariant derivatives (associated to the Levi-Civita connection). This rule is called *minimal coupling* in analogy with a similar rule for charged fields in electrodynamics.
3. Replace coordinate basis indices μ, ν etc (referring to an inertial frame) with abstract indices a, b etc.

Examples. Let x^μ denote the coordinates of an inertial frame, and $\eta^{\mu\nu}$ the inverse Minkowski metric (which has the same components as $\eta_{\mu\nu}$).

1. The simplest Lorentz invariant field equation is the wave equation for a scalar field Φ

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\Phi = 0. \tag{5.1}$$

   Follow the rules above to obtain the wave equation in a general spacetime:

$$g^{ab}\nabla_a\nabla_b\Phi = 0, \qquad \text{or} \qquad \nabla^a\nabla_a\Phi = 0 \qquad \text{or} \qquad \Phi_{;a}{}^a = 0. \tag{5.2}$$

   A simple generalization of this equation is the *Klein-Gordon equation* describing a scalar field of mass m:

$$\nabla^a\nabla_a\Phi - m^2\Phi = 0. \tag{5.3}$$

2. In special relativity, the electric and magnetic fields are combined into an antisymmetric tensor $F_{\mu\nu}$. The electric and magnetic fields in an inertial frame are obtained by the rule (i, j, k take values from 1 to 3) $F_{0i} = -E_i$ and $F_{ij} = \epsilon_{ijk} B_k$. The (source-free) Maxwell equations take the covariant form

$$\eta^{\mu\nu} \partial_\mu F_{\nu\rho} = 0, \qquad \partial_{[\mu} F_{\nu\rho]} = 0. \tag{5.4}$$

Hence in a curved spacetime, the electromagnetic field is described by an antisymmetric tensor F_{ab} satisfying

$$g^{ab} \nabla_a F_{bc} = 0, \qquad \nabla_{[a} F_{bc]} = 0. \tag{5.5}$$

The Lorentz force law for a particle of charge q and mass m in Minkowski spacetime is

$$\frac{d^2 x^\mu}{d\tau^2} = \frac{q}{m} \eta^{\mu\nu} F_{\nu\rho} \frac{dx^\rho}{d\tau} \tag{5.6}$$

where τ is proper time. We saw previously that the LHS can be rewritten as $u^\nu \partial_\nu u^\mu$ where $u^\mu = dx^\mu / d\tau$ is the 4-velocity. Now following the rules above gives the generally covariant equation

$$u^b \nabla_b u^a = \frac{q}{m} g^{ab} F_{bc} u^c = \frac{q}{m} F^a{}_b u^b. \tag{5.7}$$

Note that this reduces to the geodesic equation when $q = 0$.

Remark. The rules above ensure that we obtain generally covariant equations. But how do we know they are the *right* equations? The Einstein equivalence principle states that, in a local inertial frame, the laws of physics should take the same form as in an inertial frame in Minkowski spacetime. But we saw above, that in a local inertial frame at p, $\Gamma^\mu_{\nu\rho}(p) = 0$ and hence (first) covariant derivatives reduce to partial derivatives at p. For example, $\nabla^\mu \nabla_\mu \Phi = g^{\mu\nu} \nabla_\mu \partial_\nu \Phi$ (in any chart) and, at p, this reduces to $\eta^{\mu\nu} \partial_\mu \partial_\nu \Phi$ in a local inertial frame at p (since the metric at p is $\eta_{\mu\nu}$). Hence all of our generally covariant equations reduce to the equations of special relativity in a local inertial frame at any given point. The Einstein equivalence principle is satisfied automatically if we use the above rules. Nevertheless, there is still some scope for ambiguity, which arises from the possibility of including terms in an equation involving the curvature of spacetime (see later). These vanish identically in Minkowski spacetime. Sometimes, such terms are fixed by mathematical consistency. However, this is not always possible: there is no reason why it should be possible to derive laws of physics in curved spacetime from those in flat spacetime. The ultimate test is comparison with observations.

5.2 Energy-momentum tensor

in GR, the curvature of spacetime is related to the energy and momentum of matter. So we need to discuss how the latter concepts are defined in GR. We shall start by discussing the energy and momentum of particles.

In special relativity, associated to any particle is a scalar called its *rest mass* (or simply its mass) m. If the particle has 4-velocity u^μ (again x^μ denote inertial frame coordinates) then its 4-momentum is

$$P^\mu = mu^\mu \tag{5.8}$$

The time component of P^μ is the particle's energy and the spatial components are its 3-momentum with respect to the inertial frame.

If an observer at some point p has 4-velocity $v^\mu(p)$ then he measures the particle's energy, when the particle is at q, to be

$$E = -\eta_{\mu\nu} v^\mu(p) P^\nu(q). \tag{5.9}$$

The way to see this is to choose an inertial frame in which, at p, the observer is at rest at the origin, so $v^\mu(p) = (1, 0, 0, 0)$ so E is just the time component of $P^\nu(q)$ in this inertial frame.

By the equivalence principle, GR should reduce to SR in a local inertial frame. Hence in GR we also associate a rest mass m to any particle and define the 4-momentum of a particle with 4-velocity u^a as

$$P^a = mu^a \tag{5.10}$$

Note that

$$g_{ab} P^a P^b = -m^2 \tag{5.11}$$

The EP implies that when the observer and particle *both* are at p then (5.9) should be valid so the observer measures the particle's energy to be

$$E = -g_{ab}(p) v^a(p) P^b(p) \tag{5.12}$$

However, an important difference between GR and SR is that there is no analogue of equation (5.9) for $p \neq q$. This is because $v^a(p)$ and $P^a(q)$ are vectors defined at different points, so they live in different tangent spaces. There is no way they can be combined to give a scalar quantity. An observer at p cannot measure the energy of a particle at q.

Now let's consider the energy and momentum of continuous distributions of matter.

Example. Consider Maxwell theory (without sources) in Minkowski spacetime. Pick an inertial frame and work in pre-relativity notation using Cartesian tensors. The electromagnetic field has energy density

$$\mathcal{E} = \frac{1}{8\pi}\left(E_i E_i + B_i B_i\right) \tag{5.13}$$

and the momentum density (or energy flux density) is given by the Poynting vector:

$$S_i = \frac{1}{4\pi}\epsilon_{ijk} E_j B_k. \tag{5.14}$$

The Maxwell equations imply that these satisfy the conservation law

$$\frac{\partial \mathcal{E}}{\partial t} + \partial_i S_i = 0. \tag{5.15}$$

The momentum flux density is described by the stress tensor:

$$t_{ij} = \frac{1}{4\pi}\left[\frac{1}{2}\left(E_k E_k + B_k B_k\right)\delta_{ij} - E_i E_j - B_i B_j\right], \tag{5.16}$$

with the conservation law

$$\frac{\partial S_i}{\partial t} + \partial_j t_{ij} = 0. \tag{5.17}$$

If a surface element has area dA and normal n_i then the force exerted on this surface by the electromagnetic field is $t_{ij} n_j dA$.

In special relativity, these three objects are combined into a single tensor, called variously the "energy-momentum tensor", the "stress tensor", the "stress-energy-momentum tensor" etc. In an inertial frame it is

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}F^{\rho\sigma}F_{\rho\sigma}\,\eta_{\mu\nu}\right) \tag{5.18}$$

where we've raised indices with $\eta^{\mu\nu}$. Note that this is a symmetric tensor. It has components $T_{00} = \mathcal{E}$, $T_{0i} = -S_i$, $T_{ij} = t_{ij}$. The conservation laws above are equivalent to the single equation

$$\partial^{\mu} T_{\mu\nu} = 0. \tag{5.19}$$

The definition of the energy-momentum tensor extends naturally to GR:

Definition. The energy-momentum tensor of a Maxwell field in a general spacetime is

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F_{b}{}^{c} - \frac{1}{4}F^{cd}F_{cd}\,g_{ab}\right) \tag{5.20}$$

Exercise (examples sheet 2). Show that Maxwell's equations imply that

$$\nabla^a T_{ab} = 0. \tag{5.21}$$

In GR (and SR) we assume that continuous matter always is described by a conserved energy-momentum tensor:

Postulate. The energy, momentum, and stresses, of matter are described by an *energy-momentum tensor*, a $(0,2)$ symmetric tensor T_{ab} that is *conserved*: $\nabla^a T_{ab} = 0$.

Remark. Let u^a be the 4-velocity of an observer $\mathcal{O}$ at p. Consider a local inertial frame (LIF) at p in which $\mathcal{O}$ is at rest. Choose an orthonormal basis at p $\{e_\mu\}$ aligned with the coordinate axes of this LIF. In such a basis, $e_0^a = u^a$. Denote the spatial basis vectors as e_i^a, $i = 1, 2, 3$. From the Einstein equivalence principle, $\mathcal{E} \equiv T_{00} = T_{ab} e_0^a e_0^b = T_{ab} u^a u^b$ is the energy density of matter at p measured by $\mathcal{O}$. Similarly, $S_i \equiv -T_{0i}$ is the momentum density and $t_{ij} \equiv T_{ij}$ the stress tensor measured by $\mathcal{O}$. The *energy-momentum current* measured by $\mathcal{O}$ is the 4-vector $j^a = -T^a{}_b u^b$, which has components $(\mathcal{E}, S_i)$ in this basis.

Remark. In an inertial frame x^μ in Minkowski spacetime, local conservation of T_{ab} is equivalent to equations of the form (5.15) and (5.17). If one integrates these over a fixed volume V in surfaces of constant $t = x^0$ then one obtains global conservation equations. For example, integrating (5.15) over V gives

$$\frac{d}{dt} \int_V \mathcal{E} = - \int_S \mathbf{S} \cdot \mathbf{n} dA \tag{5.22}$$

where the surface S (with outward unit normal $\mathbf{n}$) bounds V. In words: the rate of increase of the energy of matter in V is equal to minus the energy flux across S. In a general curved spacetime, such an interpretation is not possible. This is because the gravitational field can do work on the matter in the spacetime. One might think that one could obtain global conservation laws in curved spacetime by introducing a definition of energy density etc for the gravitational field. This is a subtle issue. The gravitational field is described by the metric g_{ab}. In Newtonian theory, the energy density of the gravitational field is $-(1/8\pi)(\nabla\Phi)^2$ so one might expect that in GR the energy density of the gravitational field should be some expression quadratic in first derivatives of g_{ab}. But we have seen that we can choose normal coordinates to make the first partial derivatives of g_{ab} vanish at any given point. Gravitational energy certainly exists but not in a local sense. For example one can define the *total* energy (i.e. the energy of matter and the gravitational field) for certain spacetimes (this will be discussed in the black holes course).

Example. A *perfect fluid* is described by a 4-velocity vector field u^a, and two scalar fields ρ and p. The energy-momentum tensor is

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} \tag{5.23}$$

ρ and p are the energy density and pressure measured by an observer co-moving with the fluid, i.e., one with 4-velocity u^a (check: $T_{ab}u^a u^b = \rho + p - p = \rho$). The equations of motion of the fluid can be derived by conservation of T_{ab}:

Exercise (examples sheet 2). Show that, for a perfect fluid, $\nabla^a T_{ab} = 0$ is equivalent to

$$u^a \nabla_a \rho + (\rho + p)\nabla_a u^a = 0, \qquad (\rho + p)u^b \nabla_b u_a = -(g_{ab} + u_a u_b)\nabla^b p \tag{5.24}$$

These are relativistic generalizations of the mass conservation equation and Euler equation of non-relativistic fluid dynamics. Note that a pressureless fluid moves on timelike geodesics. This makes sense physically: zero pressure implies that the fluid particles are non-interacting and hence behave like free particles.

Chapter 6

Curvature

6.1 Parallel transport

On a general manifold there is no way of comparing tensors at different points. For example, we can't say whether a vector at p is the same as a vector at q. However, with a connection we can define a notion of "a tensor that doesn't change along a curve":

Definition. Let X^a be the tangent to a curve. A tensor field T is *parallelly transported along the curve* if $\nabla_X T = 0$.

Remarks.

1. Sometimes we say "parallelly propagated" instead of "parallelly transported".

2. A geodesic is a curve whose tangent vector is parallelly transported along the curve.

3. Let p be a point on a curve. If we specify T at p then the above equation determines T uniquely everywhere along the curve. For example, consider a $(1,1)$ tensor. Introduce a chart in a neighbourhood of p. Let t be the parameter along the curve. In a coordinate chart, $X^\mu = dx^\mu/dt$ so $\nabla_X T = 0$ gives

$$\begin{aligned} 0 &= X^\sigma T^\mu{}_{\nu;\sigma} = X^\sigma T^\mu{}_{\nu,\sigma} + \Gamma^\mu_{\rho\sigma} T^\rho{}_\nu X^\sigma - \Gamma^\rho_{\nu\sigma} T^\mu{}_\rho X^\sigma \\ &= \frac{dT^\mu{}_\nu}{dt} + \Gamma^\mu_{\rho\sigma} T^\rho{}_\nu X^\sigma - \Gamma^\rho_{\nu\sigma} T^\mu{}_\rho X^\sigma \end{aligned} \tag{6.1}$$

Standard ODE theory guarantees a unique solution given initial values for the components $T^\mu{}_\nu$.

4. If q is some other point on the curve then parallel transport along a curve from p to q determines an isomorphism between tensors at p and tensors at q.

Consider Euclidean space or Minkowski spacetime with the Levi-Civita connection, and use Cartesian/inertial frame coordinates so the Christoffel symbols vanish everywhere. Then a tensor is parallelly transported along a curve iff its components are constant along the curve. Hence if we have two different curves from p to q then the result of parallelly transporting T from p to q is independent of which curve we choose. However, in a general spacetime this is no longer true: parallel transport is path-dependent. The path-dependence of parallel transport is measured by the *Riemann curvature tensor*. For Euclidean space or Minkowski spacetime, the Riemann tensor (of the Levi-Civita connection) vanishes and we say that the spacetime is *flat*.

6.2 The Riemann tensor

We shall return to the path-dependence of parallel transport below. First we define the Riemann tensor is as follows:

Definition. The *Riemann curvature tensor* $R^a{}_{bcd}$ of a connection ∇ is defined by $R^a{}_{bcd}Z^bX^cY^d = (R(X,Y)Z)^a$, where X, Y, Z are vector fields and $R(X,Y)Z$ is the vector field

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{6.2}$$

To demonstrate that this defines a tensor, we need to show that it is linear in X, Y, Z. The symmetry $R(X,Y)Z = -R(Y,X)Z$ implies that we need only check linearity in X and Z. The non-trivial part is to check what happens if we multiply X or Z by a function f:

$$\begin{aligned} R(fX,Y)Z &= \nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]}Z \\ &= f\nabla_X \nabla_Y Z - \nabla_Y (f\nabla_X Z) - \nabla_{f[X,Y]-Y(f)X} Z \\ &= f\nabla_X\nabla_Y Z - f\nabla_Y \nabla_X Z - Y(f)\nabla_X Z - \nabla_{f[X,Y]}Z + \nabla_{Y(f)X}Z \\ &= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z - f\nabla_{[X,Y]}Z + Y(f)\nabla_X Z \\ &= fR(X,Y)Z \end{aligned} \tag{6.3}$$

$$\begin{aligned} R(X,Y)(fZ) &= \nabla_X\nabla_Y(fZ) - \nabla_Y\nabla_X(fZ) - \nabla_{[X,Y]}(fZ) \\ &= \nabla_X(f\nabla_Y Z + Y(f)Z) - \nabla_Y(f\nabla_X Z + X(f)Z) \\ &\quad -f\nabla_{[X,Y]}Z - [X,Y](f)Z \\ &= f\nabla_X\nabla_Y Z + X(f)\nabla_Y Z + Y(f)\nabla_X Z + X(Y(f))Z \end{aligned}$$

$$\begin{aligned} & -f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z - X(f)\nabla_Y Z - Y(X(f))Z \\ & -f\nabla_{[X,Y]}Z - [X,Y](f)Z \\ = & \; fR(X,Y)Z \end{aligned} \tag{6.4}$$

It follows that our definition does indeed define a tensor. Let's calculate its components in a *coordinate* basis $\{e_\mu = \partial/\partial x^\mu\}$ (so $[e_\mu, e_\nu] = 0$). Use the notation $\nabla_\mu \equiv \nabla_{e_\mu}$,

$$\begin{aligned} R(e_\rho, e_\sigma)e_\nu &= \nabla_\rho\nabla_\sigma e_\nu - \nabla_\sigma\nabla_\rho e_\nu \\ &= \nabla_\rho(\Gamma^\tau_{\nu\sigma}e_\tau) - \nabla_\sigma(\Gamma^\tau_{\nu\rho}e_\tau) \\ &= \partial_\rho\Gamma^\mu_{\nu\sigma}e_\mu + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho}e_\mu - \partial_\sigma\Gamma^\mu_{\nu\rho}e_\mu - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma}e_\mu \end{aligned} \tag{6.5}$$

and hence, in a coordinate basis,

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho} - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma} \tag{6.6}$$

Remark. It follows that the Riemann tensor vanishes for the Levi-Civita connection in Euclidean space or Minkowski spacetime (since one can choose coordinates for which the Christoffel symbols vanish everywhere).

The following contraction of the Riemann tensor plays an important role in GR:

Definition. The *Ricci curvature tensor* is the $(0,2)$ tensor defined by

$$R_{ab} = R^c{}_{acb} \tag{6.7}$$

We saw earlier that, with vanishing torsion, the second covariant derivatives of a function commute. The same is not true of covariant derivatives of tensor fields. The failure to commute arises from the Riemann tensor:

Exercise. Let ∇ be a torsion-free connection. Prove the *Ricci identity*:

$$\nabla_c\nabla_d Z^a - \nabla_d\nabla_c Z^a = R^a{}_{bcd}Z^b \tag{6.8}$$

Hint. Show that the equation is true when multiplied by arbitrary vector fields X^c and Y^d.

6.3 Parallel transport again

Now we return to the relation between the Riemann tensor and the path-dependence of parallel transport. Let X and Y be vector fields that are linearly independent everywhere, with $[X,Y] = 0$. Earlier we saw that we can choose a coordinate chart

$(s, t, \ldots)$ such that $X = \partial/\partial s$ and $Y = \partial/\partial t$. Let $p \in M$ and choose the coordinate chart such that p has coordinates $(0, \ldots, 0)$. Let q, r, u be the point with coordinates $(\delta s, 0, 0, \ldots)$, $(\delta s, \delta t, 0, \ldots)$, $(0, \delta t, 0, \ldots)$ respectively, where δs and δt are small. We can connect p and q with a curve along which only s varies, with tangent X. Similarly, q and r can be connected by a curve with tangent Y. p and u can be connected by a curve with tangent Y, and u and r can be connected by a curve with tangent X. The result is a small quadrilateral (Fig. 6.1).

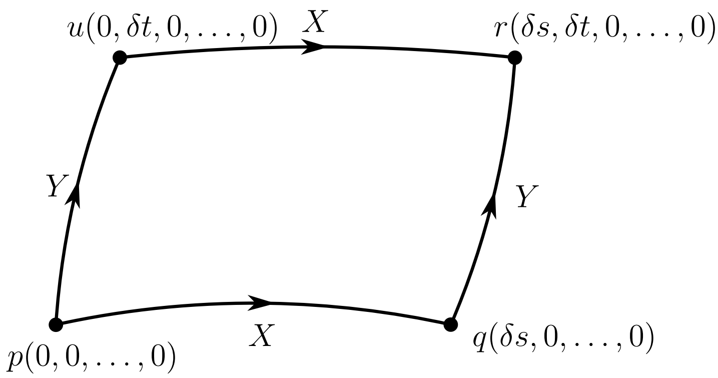


Figure 6.1: Parallel transport

Now let $Z_p \in T_p(M)$. Parallel transport Z_p along pqr to obtain a vector $Z_r \in T_r(M)$. Parallel transport Z_p along pur to obtain a vector $Z'_r \in T_r(M)$. We shall calculate the difference $Z'_r - Z_r$ for a torsion-free connection.

It is convenient to introduce a new coordinate chart: normal coordinates at p. Henceforth, indices $\mu, \nu, \ldots$ will refer to this chart. s and t will now be used as *parameters* along the curves with tangent X and Y respectively.

pq is a curve with tangent vector X and parameter s. Along pq, Z is parallely transported: $\nabla_X Z = 0$ so $dZ^\mu/ds = -\Gamma^\mu_{\nu\rho} Z^\nu X^\rho$ and hence $d^2 Z^\mu/ds^2 = -(\Gamma^\mu_{\nu\rho} Z^\nu X^\rho)_{,\sigma} X^\sigma$. Now Taylor's theorem gives

$$\begin{aligned} Z^\mu_q &= Z^\mu_p + \left(\frac{dZ^\mu}{ds}\right)_p \delta s + \frac{1}{2}\left(\frac{d^2 Z^\mu}{ds^2}\right)_p \delta s^2 + \mathcal{O}(\delta s^3) \\ &= Z^\mu_p - \frac{1}{2}\left(\Gamma^\mu_{\nu\rho,\sigma} Z^\nu X^\rho X^\sigma\right)_p \delta s^2 + \mathcal{O}(\delta s^3) \end{aligned} \tag{6.9}$$

where we have used $\Gamma^\mu_{\nu\rho}(p) = 0$ in normal coordinates at p (assuming a torsion-free connection). Now consider parallel transport along qr to obtain

$$Z^\mu_r = Z^\mu_q + \left(\frac{dZ^\mu}{dt}\right)_q \delta t + \frac{1}{2}\left(\frac{d^2 Z^\mu}{dt^2}\right)_q \delta t^2 + \mathcal{O}(\delta t^3)$$

$$
\begin{aligned}
&= Z_q^\mu - \left(\Gamma^\mu_{\nu\rho} Z^\nu Y^\rho\right)_q \delta t - \frac{1}{2}\left((\Gamma^\mu_{\nu\rho} Z^\nu Y^\rho)_{,\sigma} Y^\sigma\right)_q \delta t^2 + \mathcal{O}(\delta t^3) \\
&= Z_q^\mu - \left[\left(\Gamma^\mu_{\nu\rho,\sigma} Z^\nu Y^\rho X^\sigma\right)_p \delta s + \mathcal{O}(\delta s^2)\right]\delta t \\
&\quad -\frac{1}{2}\left[\left((\Gamma^\mu_{\nu\rho,\sigma} Z^\nu Y^\rho Y^\sigma\right)_p + \mathcal{O}(\delta s)\right]\delta t^2 + \mathcal{O}(\delta t^3) \\
&= Z_p^\mu - \frac{1}{2}\left(\Gamma^\mu_{\nu\rho,\sigma}\right)_p \left[Z^\nu\left(X^\rho X^\sigma \delta s^2 + Y^\rho Y^\sigma \delta t^2 + 2Y^\rho X^\sigma \delta s \delta t\right)\right]_p + \mathcal{O}(\delta^3)
\end{aligned}
\tag{6.10}
$$

Here we assume that δs and δt both are $\mathcal{O}(\delta)$ (i.e. $\delta s = a\delta$ for some non-zero constant a and similarly for δt). Now consider parallel transport along pur. The result can be obtained from the above expression simply by interchanging X with Y and s with t. Hence we have

$$
\begin{aligned}
\Delta Z_r^\mu \equiv Z_r'^\mu - Z_r^\mu &= \left[\Gamma^\mu_{\nu\rho,\sigma} Z^\nu\left(Y^\rho X^\sigma - X^\rho Y^\sigma\right)\right]_p \delta s \delta t + \mathcal{O}(\delta^3) \\
&= \left[\left(\Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\nu\rho,\sigma}\right) Z^\nu X^\rho Y^\sigma\right]_p \delta s \delta t + \mathcal{O}(\delta^3) \\
&= \left(R^\mu{}_{\nu\rho\sigma} Z^\nu X^\rho Y^\sigma\right)_p \delta s \delta t + \mathcal{O}(\delta^3) \\
&= \left(R^\mu{}_{\nu\rho\sigma} Z^\nu X^\rho Y^\sigma\right)_r \delta s \delta t + \mathcal{O}(\delta^3)
\end{aligned}
\tag{6.11}
$$

where we used the expression (6.6) for the Riemann tensor components (remember that $\Gamma^\mu_{\nu\rho}(p) = 0$). In the final equality we used that quantities at p and r differ by $\mathcal{O}(\delta)$. We have derived this result in a coordinate basis defined using normal coordinates at p. But now both sides involve tensors at r. Hence our equation is basis-independent so we can write

$$
\left(R^a{}_{bcd} Z^b X^c Y^d\right)_r = \lim_{\delta \to 0} \frac{\Delta Z_r^a}{\delta s \delta t}
\tag{6.12}
$$

The Riemann tensor measures the path-dependence of parallel transport.

Remark. We considered parallel transport along two different curves from p to r. However, we can reinterpret the result as describing the effect of parallel transport of a vector Z_r^a around the closed curve $rqpur$ to give the vector $Z_r'^a$. Hence ΔZ_r^a measures the change in Z_r^a when parallel transported around a closed curve.

6.4 Symmetries of the Riemann tensor

From its definition, we have the symmetry $R^a{}_{bcd} = -R^a{}_{bdc}$, equivalently:

$$
R^a{}_{b(cd)} = 0.
\tag{6.13}
$$

Proposition. If ∇ is torsion-free then

$$R^a{}_{[bcd]} = 0. \tag{6.14}$$

Proof. Let $p \in M$ and choose normal coordinates at p. Vanishing torsion implies $\Gamma^\mu_{\nu\rho}(p) = 0$ and $\Gamma^\mu_{[\nu\rho]} = 0$ everywhere. We have $R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho}$ at p. Antisymmetrizing on $\nu\rho\sigma$ now gives $R^\mu{}_{[\nu\rho\sigma]} = 0$ at p in the coordinate basis defined using normal coordinates at p. But if the components of a tensor vanish in one basis then they vanish in any basis. This proves the result at p. However, p is arbitrary so the result holds everywhere.

Proposition. (Bianchi identity). If ∇ is torsion-free then

$$R^a{}_{b[cd;e]} = 0 \tag{6.15}$$

Proof. Use normal coordinate at p again. At p,

$$R^\mu{}_{\nu\rho\sigma;\tau} = \partial_\tau R^\mu{}_{\nu\rho\sigma} \tag{6.16}$$

In normal coordinates at p, $\partial R = \partial\partial\Gamma - \Gamma\partial\Gamma$ and the latter terms vanish at p, we only need to worry about the former:

$$R^\mu{}_{\nu\rho\sigma;\tau} = \partial_\tau \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\tau \partial_\sigma \Gamma^\mu_{\nu\rho} \qquad \text{at } p \tag{6.17}$$

Antisymmetrizing gives $R^\mu{}_{\nu[\rho\sigma;\tau]} = 0$ at p in this basis. But again, if this is true in one basis then it is true in any basis. Furthermore, p is arbitrary. The result follows.

6.5 Geodesic deviation

Remark. In Euclidean space, or in Minkowski spacetime, initially parallel geodesics remain parallel forever. On a general manifold we have no notion of "parallel". However, we can study whether nearby geodesics move together or apart. In particular, we can quantify their "relative acceleration".

Definition. Let M be a manifold with a connection ∇. A *1-parameter family of geodesics* is a map $\gamma : I \times I' \to M$ where I and I' both are open intervals in $\mathbb{R}$, such that (i) for fixed s, $\gamma(s,t)$ is a geodesic with affine parameter t (so s is the parameter that labels the geodesic); (ii) the map $(s,t) \mapsto \gamma(s,t)$ is smooth and one-to-one with a smooth inverse. This implies that the family of geodesics forms a 2d surface $\Sigma \subset M$.

Let T be the tangent vector to the geodesics and S to be the vector tangent to the curves of constant t, which are parameterized by s (see Fig. 6.2). In a chart x^μ, the geodesics are specified by $x^\mu(s,t)$ with $S^\mu = \partial x^\mu / \partial s$. Hence

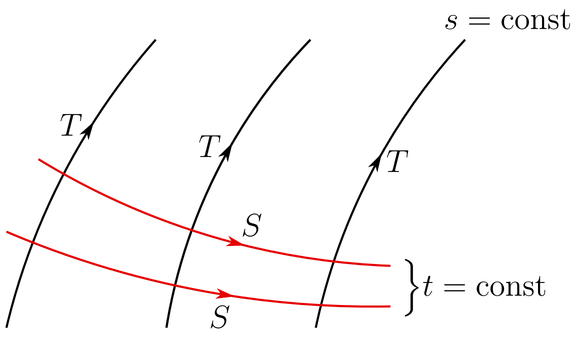


Figure 6.2: 1-parameter family of geodesics

$x^\mu(s+\delta s,t) = x^\mu(s,t) + \delta s S^\mu(s,t) + \mathcal{O}(\delta s^2)$. Therefore $(\delta s)S^a$ points from one geodesic to an infinitesimally nearby one in the family. We call S^a a *deviation vector*.

On the surface Σ we can use s and t as coordinates. We can extend these to coordinates $(s,t,\ldots)$ defined in a neighbourhood of Σ. This gives a coordinate chart in which $S = \partial/\partial s$ and $T = \partial/\partial t$ on Σ. We can now use these equations to extend S and T to a neighbourhood of the surface. S and T are now vector fields satisfying

$$[S,T] = 0 \tag{6.18}$$

Remark. If we fix attention on a particular geodesic then $\nabla_T(\delta s S) = \delta s \nabla_T S$ can be regarded as the rate of change of the relative position of an infinitesimally nearby geodesic in the family i.e., as the "relative velocity" of an infinitesimally nearby geodesic. We can define the "relative acceleration" of an infinitesimally nearby geodesic in the family as $\delta s \nabla_T \nabla_T S$. The word "relative" is important: the acceleration of a curve with tangent T is $\nabla_T T$, which vanishes here (as the curves are geodesics).

Proposition. If ∇ has vanishing torsion then

$$\nabla_T \nabla_T S = R(T,S)T \tag{6.19}$$

Proof. Vanishing torsion gives $\nabla_T S - \nabla_S T = [T,S] = 0$. Hence

$$\nabla_T \nabla_T S = \nabla_T \nabla_S T = \nabla_S \nabla_T T + R(T,S)T, \tag{6.20}$$

where we used the definition of the Riemann tensor. But $\nabla_T T = 0$ because T is tangent to (affinely parameterized) geodesics.

Remark. This result is known as the *geodesic deviation equation.* In abstract index notation it is:

$$T^c\nabla_c(T^b\nabla_b S^a) = R^a{}_{bcd}T^bT^cS^d \tag{6.21}$$

This equation shows that curvature results in relative acceleration of geodesics. It also provides another method of measuring $R^a{}_{bcd}$: at any point p we can pick our 1-parameter family of geodesics such that T and S are arbitrary. Hence by measuring the LHS above we can determine $R^a{}_{(bc)d}$. From this we can determine $R^a{}_{bcd}$:

Exercise. Show that, for a torsion-free connection,

$$R^a{}_{bcd} = \frac{2}{3}\left(R^a{}_{(bc)d} - R^a{}_{(bd)c}\right) \tag{6.22}$$

Remarks.

1. Note that the relative acceleration vanishes for *all* families of geodesics if, and only if, $R^a{}_{bcd} = 0$.

2. In GR, free particles follow geodesics of the Levi-Civita connection. Geodesic deviation is the tendency of freely falling particles to move together or apart. We have already met this phenomenon: it arises from *tidal forces.* Hence the Riemann tensor is the quantity that measures tidal forces.

6.6 Curvature of the Levi-Civita connection

Remark. From now on, we shall restrict attention to a manifold with metric, and use the Levi-Civita connection. The Riemann tensor then enjoys additional symmetries. Note that we can lower an index with the metric to define R_{abcd}.

Proposition. The Riemann tensor satisfies

$$R_{abcd} = R_{cdab}, \qquad R_{(ab)cd} = 0. \tag{6.23}$$

Proof. The second identity follows from the first and the antisymmetry of the Riemann tensor. To prove the first, introduce normal coordinates at p, so $\partial_\mu g_{\nu\rho} = 0$ at p. Then, at p,

$$0 = \partial_\mu \delta^\nu_\rho = \partial_\mu\left(g^{\nu\sigma}g_{\sigma\rho}\right) = g_{\sigma\rho}\partial_\mu g^{\nu\sigma}. \tag{6.24}$$

Multiplying by the inverse metric gives $\partial_\mu g^{\nu\rho} = 0$ at p. Using this, we have

$$\partial_\rho \Gamma^\tau_{\nu\sigma} = \frac{1}{2}g^{\tau\mu}\left(g_{\mu\nu,\sigma\rho} + g_{\mu\sigma,\nu\rho} - g_{\nu\sigma,\mu\rho}\right) \qquad \text{at } p \tag{6.25}$$

And hence (as $\Gamma^\mu_{\nu\rho} = 0$ at p)

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(g_{\mu\sigma,\nu\rho} + g_{\nu\rho,\mu\sigma} - g_{\nu\sigma,\mu\rho} - g_{\mu\rho,\nu\sigma}\right) \qquad \text{at } p \tag{6.26}$$

This satisfies $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$ at p using the symmetry of the metric and the fact that partial derivatives commute. This establishes the identity in normal coordinates, but this is a tensor equation and hence valid in any basis. Furthermore p is arbitrary so the identity holds everywhere.

Proposition. The Ricci tensor is symmetric:

$$R_{ab} = R_{ba} \tag{6.27}$$

Proof. $R_{ab} = g^{cd}R_{dacb} = g^{cd}R_{cbda} = R^c{}_{bca} = R_{ba}$ where we used the first identity above in the second equality.

Definition. The *Ricci scalar* is

$$R = g^{ab}R_{ab} \tag{6.28}$$

Definition. The *Einstein tensor* is the symmetric $(0,2)$ tensor defined by

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} \tag{6.29}$$

Proposition. The Einstein tensor satisfies the *contracted Bianchi identity*:

$$\nabla^a G_{ab} = 0 \tag{6.30}$$

which can also be written as

$$\nabla^a R_{ab} - \frac{1}{2}\nabla_b R = 0 \tag{6.31}$$

Proof. Examples sheet 2.

6.7 Einstein's equation

Postulates of General Relativity.

1. Spacetime is a 4d Lorentzian manifold equipped with the Levi-Civita connection.

2. Free particles follow timelike or null geodesics.

3. The energy, momentum, and stresses of matter are described by a symmetric tensor T_{ab} that is conserved: $\nabla^a T_{ab} = 0$.

4. The curvature of spacetime is related to the energy-momentum tensor of matter by the *Einstein equation* (1915)

$$G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G \, T_{ab} \tag{6.32}$$

   where G is Newton's constant.

We have discussed points 1-3 above. It remains to discuss the Einstein equation. We can motivate this as follows. In GR, the gravitational field is described by the curvature of spacetime. Since the energy of matter should be responsible for gravitation, we expect *some* relationship between curvature and the energy-momentum tensor. The simplest possibility is a linear relationship, i.e., a curvature tensor is proportional to T_{ab}. Since T_{ab} is symmetric, it is natural to expect the Ricci tensor to be the relevant curvature tensor.

Einstein's first guess for the field equation of GR was $R_{ab} = C T_{ab}$ for some constant C. This does not work for the following reason. The RHS is conserved hence this equation implies $\nabla^a R_{ab} = 0$. But then from the contracted Bianchi identity we get $\nabla_a R = 0$. Taking the trace of the equation gives $R = CT$ (where $T = T^a{}_a$) and hence we must have $\nabla_a T = 0$, i.e., T is constant. But, T vanishes in empty space and is usually non-zero inside matter. Hence this is unsatisfactory.

The solution to this problem is obvious once one knows of the contracted Bianchi identity. Take G_{ab}, rather than R_{ab}, to be proportional to T_{ab}. The coefficient of proportionality on the RHS of Einstein's equation is fixed by demanding that the equation reduces to Newton's law of gravitation when the gravitational field is weak and the matter is moving non-relativistically. We will show this later.

Remarks.

1. In vacuum, $T_{ab} = 0$ so Einstein's equation gives $G_{ab} = 0$. Contracting indices gives $R = 0$. Hence the *vacuum Einstein equation* can be written as

$$R_{ab} = 0 \tag{6.33}$$

2. The "geodesic postulate" of GR is redundant. Using conservation of the energy-momentum tensor it can be shown that a distribution of matter that is small (compared to the scale on which the spacetime metric varies), and sufficiently weak (so that its gravitational effect is small), must follow a geodesic. (See examples sheet 4 for the case of a point particle.)

3. The Einstein equation is a set of non-linear, second order, coupled, partial differential equations for the components of the metric $g_{\mu\nu}$. Very few physically interesting explicit solutions are known so one has to develop other methods to solve the equation, e.g., numerical integration.

4. How unique is the Einstein equation? Is there any tensor, other than G_{ab} that we could have put on the LHS? The answer is supplied by:

Theorem (Lovelock 1972). Let H_{ab} be a symmetric tensor such that (i) in any coordinate chart, at any point, $H_{\mu\nu}$ is a function of $g_{\mu\nu}$, $g_{\mu\nu,\rho}$ and $g_{\mu\nu,\rho\sigma}$ at that point; (ii) $\nabla^a H_{ab} = 0$; (iii) either spacetime is four-dimensional or $H_{\mu\nu}$ depends linearly on $g_{\mu\nu,\rho\sigma}$. Then there exist constants α and β such that

$$H_{ab} = \alpha G_{ab} + \beta g_{ab} \tag{6.34}$$

Hence (as Einstein realized) there is the freedom to add a constant multiple of g_{ab} to the LHS of Einstein's equation, giving

$$G_{ab} + \Lambda g_{ab} = 8\pi G\, T_{ab} \tag{6.35}$$

Λ is called the *cosmological constant*. This no longer reduces to Newtonian theory for slow motion in a weak field but the deviation from Newtonian theory is unobservable if Λ is sufficiently small. Note that $|\Lambda|^{-1/2}$ has the dimensions of length. The effects of Λ are negligible on lengths or times small compared to this quantity. Astronomical observations suggest that there is indeed a very small positive cosmological constant: $\Lambda^{-1/2} \sim 10^9$ light years, the same order of magnitude as the size of the observable Universe. Hence the effects of the cosmological constant are negligible except on cosmological length scales. Therefore we can set $\Lambda = 0$ unless we discuss cosmology.

Note that we can move the cosmological constant term to the RHS of the Einstein equation, and regard it as the energy-momentum tensor of a perfect fluid with $\rho = -p = \Lambda/(8\pi G)$. Hence the cosmological constant is sometimes referred to as *dark energy* or *vacuum energy*. It is a great mystery why it is so small because arguments from quantum field theory suggest that it should be 10^{120} times larger. This is the *cosmological constant problem*. One (controversial) proposed solution of this problem invokes the *anthropic principle*, which posits the existence of many possible universes with different values for constants such as Λ. If Λ was very different from its observed value then galaxies never would have formed and hence we would not be here.

Remark. We have explicitly written Newton's constant G throughout this section. Henceforth we shall choose units so that $G = c = 1$.

Chapter 7

Diffeomorphisms and Lie derivative

7.1 Maps between manifolds

Definition. Let M, N be differentiable manifolds of dimension m, n respectively. A function $\phi : M \to N$ is *smooth* if, and only if, $\psi_A \circ \phi \circ \psi_\alpha^{-1}$ is smooth for all charts ψ_α of M and all charts ψ_A of N (note that this is a map from a subset of $\mathbb{R}^m$ to a subset of $\mathbb{R}^n$).

If we have such a map then we can "pull-back" a function on N to define a function on M:

Definition. Let $\phi : M \to N$ and $f : N \to \mathbb{R}$ be smooth functions. The *pull-back* of f by ϕ is the function $\phi^*(f) : M \to \mathbb{R}$ defined by $\phi^*(f) = f \circ \phi$, i.e., $\phi^*(f)(p) = f(\phi(p))$.

Furthermore, ϕ allows us to "push-forward" a curve λ in M to a curve $\phi \circ \lambda$ in N. Hence we can push-forward vectors from M to N (Figs. 7.1, 7.2)

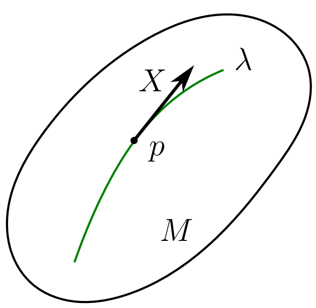


Figure 7.1: A curve in M

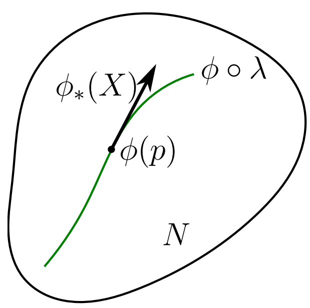


Figure 7.2: The curve in N

Definition. Let $\phi : M \to N$ be smooth. Let $p \in M$ and $X \in T_p(M)$. The *push-forward* of X with respect to ϕ is the vector $\phi_*(X) \in T_{\phi(p)}(N)$ defined as follows. Let λ be a smooth curve in M passing through p with tangent X at p. Then $\phi_*(X)$ is the tangent vector to the curve $\phi \circ \lambda$ in N at the point $\phi(p)$.

Lemma. Let $f : N \to \mathbb{R}$. Then $(\phi_*(X))(f) = X(\phi^*(f))$.

Proof. Wlog $\lambda(0) = p$.

$$\begin{aligned}(\phi_*(X))(f) &= \left[\frac{d}{dt}(f \circ (\phi \circ \lambda))(t)\right]_{t=0} \\ &= \left[\frac{d}{dt}((f \circ \phi) \circ \lambda)(t)\right]_{t=0} \\ &= X(\phi^*(f))\end{aligned} \tag{7.1}$$

Exercise. Let x^μ be coordinates on M and y^α be coordinates on N (we use different indices α, β etc for N because N is a different manifold which might not have the same dimension as M). Then we can regard ϕ as defining a map $y^\alpha(x^\mu)$. Show that the components of $\phi_*(X)$ are related to those of X by

$$(\phi_*(X))^\alpha = \left(\frac{\partial y^\alpha}{\partial x^\mu}\right)_p X^\mu \tag{7.2}$$

The map on covectors works in the opposite direction:

Definition. Let $\phi : M \to N$ be smooth. Let $p \in M$ and $\eta \in T^*_{\phi(p)}(N)$. The *pull-back* of η with respect to ϕ is $\phi^*(\eta) \in T^*_p(M)$ defined by $(\phi^*(\eta))(X) = \eta(\phi_*(X))$ for any $X \in T_p(M)$.

Lemma. Let $f : N \to \mathbb{R}$. Then $\phi^*(df) = d(\phi^*(f))$.

Proof. Let $X \in T_p(M)$. Then

$$(\phi^*(df))(X) = (df)(\phi_*(X)) = (\phi_*(X))(f) = X(\phi^*(f)) = (d(\phi^*(f)))(X) \tag{7.3}$$

The first equality is the definition of ϕ^*, the second is the definition of df, the third is the previous Lemma and the fourth is the definition of $d(\phi^*(f))$. Since X is arbitrary, the result follows.

Exercise. Use coordinates x^μ and y^α as before. Show that the components of $\phi^*(\eta)$ are related to the components of η by

$$(\phi^*(\eta))_\mu = \left(\frac{\partial y^\alpha}{\partial x^\mu}\right)_p \eta_\alpha \tag{7.4}$$

Remarks.

1. In all of the above, the point p was arbitrary so push-forward and pull-back can be applied to vector and covector *fields*, respectively.

2. The pull-back can be extended to a tensor S of type $(0,s)$ by defining $(\phi^*(S))(X_1, \dots X_s) = S(\phi_*(X_1), \dots \phi_*(X_s))$ where $X_1, \dots, X_s \in T_p(M)$. Similarly, one can push-forward a tensor of type $(r,0)$ by defining $\phi_*(T)(\eta_1, \dots, \eta_r) = T(\phi^*(\eta_1), \dots, \phi^*(\eta_r))$ where $\eta_1, \dots, \eta_r \in T^*_p(N)$. The components of these tensors in a coordinate basis are given by

$$(\phi^*(S))_{\mu_1 \dots \mu_s} = \left(\frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \right)_p \cdots \left(\frac{\partial y^{\alpha_s}}{\partial x^{\mu_s}} \right)_p S_{\alpha_1 \dots \alpha_s} \tag{7.5}$$

$$(\phi_*(T))^{\alpha_1 \dots \alpha_r} = \left(\frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \right)_p \cdots \left(\frac{\partial y^{\alpha_r}}{\partial x^{\mu_r}} \right)_p T^{\mu_1 \dots \mu_r} \tag{7.6}$$

Example. The embedding of S^2 into Euclidean space. Let $M = S^2$ and $N = \mathbb{R}^3$. Define $\phi : M \to N$ as the map which sends the point on S^2 with spherical polar coordinates $x^\mu = (\theta, \phi)$ to the point $y^\alpha = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \in \mathbb{R}^3$. Consider the Euclidean metric g on $\mathbb{R}^3$, whose components are the identity matrix $\delta_{\alpha\beta}$. Pulling this back to S^2 using (7.5) gives $(\phi^* g)_{\mu\nu} = \mathrm{diag}(1, \sin^2\theta)$ (check!), the unit round metric on S^2.

7.2 Diffeomorphisms, Lie Derivative

Definition. A map $\phi : M \to N$ is a *diffeomorphism* iff it 1-1 and onto, smooth, and has a smooth inverse.

Remark. This implies that M and N have the same dimension. In fact, M and N have identical manifold structure.

With a diffeomorphism, we can extend our definitions of push-forward and pull-back so that they apply for any type of tensor:

Definition. Let $\phi : M \to N$ be a diffeomorphism and T a tensor of type (r,s) on M. Then the *push-forward* of T is a tensor $\phi_*(T)$ of type (r,s) on N defined by (for arbitrary $\eta_i \in T^*_{\phi(p)}(N)$, $X_i \in T_{\phi(p)}(N)$)

$$\phi_*(T)(\eta_1, \dots, \eta_r, X_1, \dots, X_s) = T(\phi^*(\eta_1), \dots, \phi^*(\eta_r), (\phi^{-1})_*(X_1), \dots, (\phi^{-1})_*(X_s)) \tag{7.7}$$

Exercises.

1. Convince yourself that push-forward commutes with the contraction and outer product operations.

2. Show that the analogue of equation (7.6) for a $(1,1)$ tensor field is

$$[(\phi_*(T))^\mu{}_\nu]_{\phi(p)} = \left(\frac{\partial y^\mu}{\partial x^\rho}\right)_p \left(\frac{\partial x^\sigma}{\partial y^\nu}\right)_p (T^\rho{}_\sigma)_p \tag{7.8}$$

(We don't need to use indices α, β etc because now M and N have the same dimension.) Generalize this result to a (r,s) tensor.

Remarks.

1. Pull-back can be defined in a similar way, with the result $\phi^* = (\phi^{-1})_*$.

2. We've taken an "active" point of view, regarding a diffeomorphism as a map taking a point p to a new point $\phi(p)$. However, there is an alternative "passive" point of view in which we consider a diffeomorphism simply as a change of chart at p. Consider a coordinate chart x^μ defined near p and another chart y^μ defined near $\phi(p)$ (Fig. 7.3). Regarding the coordinates y^μ as functions on N, we can pull them back to define corresponding coordinates, which we also call y^μ, on M. So now we have two coordinate systems defined near p. The components of tensors at p in the new coordinate basis are given by the tensor transformation law, which is exactly the RHS of (7.8).

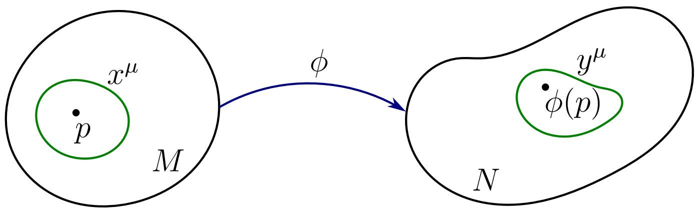


Figure 7.3: Active versus passive diffeomorphism.

Definition. Let $\phi : M \to N$ be a diffeomorphism. Let ∇ be a covariant derivative on M. The push-forward of ∇ is a covariant derivative $\tilde{\nabla}$ on N defined by

$$\tilde{\nabla}_X T = \phi_* \left(\nabla_{\phi^*(X)} \left(\phi^*(T) \right) \right) \tag{7.9}$$

where X is a vector field and T a tensor field on N. (In words: pull-back X and T to M, evaluate the covariant derivative there and then push-forward the result to N.)

Exercises (examples sheet 3).

1. Check that this satisfies the properties of a covariant derivative.

2. Show that the Riemann tensor of $\tilde{\nabla}$ is the push-forward of the Riemann tensor of ∇.

3. Let ∇ be the Levi-Civita connection defined by a metric g on M. Show that $\tilde{\nabla}$ is the Levi-Civita connection defined by the metric $\phi_*(g)$ on N.

Remark In GR we describe physics with a manifold M on which certain tensor fields e.g. the metric g, Maxwell field F etc. are defined. If $\phi : M \to N$ is a diffeomorphism then there is no way of distinguishing $(M, g, F, \ldots)$ from $(N, \phi_*(g), \phi_*(F), \ldots)$; they give equivalent descriptions of physics. For example if the metric g on M has components $g_{\mu\nu}$ in a basis $\{e_\mu\}$ for $T_p(M)$ then the metric $\phi_*(g)$ has the same components $g_{\mu\nu}$ in the basis $\{\phi_*(e_\mu)\}$ for $T_{\phi(p)}(N)$. If we set $N = M$ this reveals that the set of tensor fields $(\phi_*(g), \phi_*(F), \ldots)$ is physically indistinguishable from $(g, F, \ldots)$. If two sets of tensor fields are not related by a diffeomorphism then they *are* physically distinguishable. It follows that diffeomorphisms are the gauge symmetry (redundancy of description) in GR.

Example. Consider three particles following geodesics of the metric g. Assume that the worldlines of particles 1 and 2 intersect at p and that the worldlines of particles 2 and 3 intersect at q. Applying a diffeomorphism $\phi : M \to M$ maps the worldlines to geodesics of $\phi_*(g)$ which intersect at the points $\phi(p)$ and $\phi(q)$. Note that $\phi(p) \neq p$ so saying "particles 1 and 2 coincide at p" is not a gauge-invariant statement. An example of a quantity that *is* gauge invariant is the proper time between the two intersections along the worldline of particle 2.

Remark. This gauge freedom raises the following puzzle. The metric tensor is symmetric and hence has 10 independent components. Consider the vacuum Einstein equation - this appears to give 10 independent equations, which looks good. But the Einstein equation should not determine the components of the metric tensor uniquely, but only up to diffeomorphisms. The resolution is that not all components of the Einstein equations are truly independent because they are related by the contracted Bianchi identity.

Note that diffeomorphisms allow us to compare tensors defined at different points via push-forward or pull-back. This leads to a notion of a tensor field possessing symmetry:

Definition. A diffeomorphism $\phi : M \to M$ is a *symmetry transformation* of a tensor field T iff $\phi_*(T) = T$ everywhere. A symmetry transformation of the metric tensor is called an *isometry.*

Definition. Let X be a vector field on a manifold M. Let ϕ_t be the map which sends a point $p \in M$ to the point parameter distance t along the integral curve of X through p (this might be defined only for small enough t). It can be shown that ϕ_t is a diffeomorphism.

Remarks.

1. Note that ϕ_0 is the identity map and $\phi_s \circ \phi_t = \phi_{s+t}$. Hence $\phi_{-t} = (\phi_t)^{-1}$. If ϕ_t is defined for all $t \in \mathbb{R}$ (in which case we say the integral curves of X are *complete*) then these diffeomorphisms form a 1-parameter abelian group.

2. Given X we've defined ϕ_t. Conversely, if one has a 1-parameter abelian group of diffeomorphisms ϕ_t (i.e. one satisfying the rules just mentioned) then through any point p one can consider the curve with parameter t given by $\phi_t(p)$. Define X to be the tangent to this curve at p. Doing this for all p defines a vector field X. The integral curves of X generate ϕ_t in the sense defined above.

3. If we use $(\phi_t)_*$ to compare tensors at different points then the parameter t controls how near the points are. In particular, in the limit $t \to 0$, we are comparing tensors at infinitesimally nearby points. This leads to the notion of a new type of derivative:

Definition. The *Lie derivative* of a tensor field T with respect to a vector field X at p is

$$(\mathcal{L}_X T)_p = \lim_{t\to 0} \frac{((\phi_{-t})_* T)_p - T_p}{t} \tag{7.10}$$

Remark. The Lie derivative wrt X is a map from (r,s) tensor fields to (r,s) tensor fields. It obeys $\mathcal{L}_X(\alpha S + \beta T) = \alpha \mathcal{L}_X S + \beta \mathcal{L}_X T$ where α and β are constants.

The easiest way to demonstrate other properties of the Lie derivative is to introduce coordinates in which the components of X are simple. Let Σ be a hypersurface that has the property that X is nowhere tangent to Σ (in particular $X \neq 0$ on Σ). Let x^i, $i = 1, 2, \ldots, n-1$ be coordinates on Σ. Now assign coordinates (t, x^i) to the point parameter distance t along the integral curve of X that starts at the point with coordinates x^i on Σ (Fig. 7.4).

This defines a coordinate chart (t, x^i) at least for small t, i.e., in a neighbourhood of Σ. Furthermore, the integral curves of X are the curves (t, x^i) with fixed x^i and parameter t. The tangent to these curves is $\partial/\partial t$ so we have constructed coordinates such that $X = \partial/\partial t$. The diffeomorphism ϕ_t is very simple: it just sends the point p with coordinates $x^\mu = (t_p, x^i_p)$ to the point $\phi_t(p)$ with coordinates $y^\mu = (t_p + t, x^i_p)$ hence $\partial y^\mu / \partial x^\nu = \delta^\mu_\nu$. The generalization of (7.8) to a (r,s) tensor then gives

$$[((\phi_{-t})_*(T))^{\mu_1,\ldots,\mu_r}{}_{\nu_1,\ldots\nu_s}]_{\phi_{-t}(q)} = [T^{\mu_1,\ldots,\mu_r}{}_{\nu_1,\ldots\nu_s}]_q \tag{7.11}$$

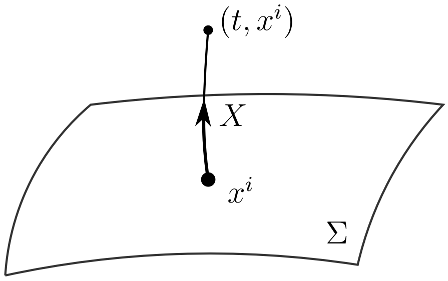


Figure 7.4: Coordinates adapted to a vector field

and setting $q = \phi_t(p)$ gives

$$[((\phi_{-t})_*(T))^{\mu_1,\dots,\mu_r}{}_{\nu_1,\dots\nu_s}]_p = [T^{\mu_1,\dots,\mu_r}{}_{\nu_1,\dots\nu_s}]_{\phi_t(p)} \tag{7.12}$$

It follows that, if p has coordinates (t_p, x^i_p) in this chart,

$$\begin{aligned}(\mathcal{L}_X T)^{\mu_1,\dots,\mu_r}{}_{\nu_1,\dots\nu_s} &= \lim_{t\to 0}\frac{1}{t}\left[T^{\mu_1,\dots,\mu_r}{}_{\nu_1,\dots\nu_s}(t_p+t, x^i_p) - T^{\mu_1,\dots,\mu_r}{}_{\nu_1,\dots\nu_s}(t_p, x^i_p)\right] \\ &= \left(\frac{\partial}{\partial t}T^{\mu_1,\dots,\mu_r}{}_{\nu_1,\dots\nu_s}\right)(t_p, x^i_p) \end{aligned}\tag{7.13}$$

So *in this chart*, the Lie derivative is simply the partial derivative with respect to the coordinate t. It follows that the Lie derivative has the following properties:

1. It obeys the Leibniz rule: $\mathcal{L}_X(S\otimes T) = (\mathcal{L}_X S)\otimes T + S\otimes \mathcal{L}_X T$.

2. It commutes with contraction.

Now let's derive a basis-independent formula for the Lie derivative. First consider a function f. In the above chart, we have $\mathcal{L}_X f = (\partial/\partial t)(f)$. However, in this chart we also have $X(f) = (\partial/\partial t)(f)$. Hence

$$\mathcal{L}_X f = X(f) \tag{7.14}$$

Both sides of this expression are scalars and hence this equation must be valid in any basis. Next consider a vector field Y. In our coordinates above we have

$$(\mathcal{L}_X Y)^\mu = \frac{\partial Y^\mu}{\partial t} \tag{7.15}$$

but we also have

$$[X, Y]^\mu = \frac{\partial Y^\mu}{\partial t} \tag{7.16}$$

If two vectors have the same components in one basis then they are equal in all bases. Hence we have the basis-independent result

$$\mathcal{L}_X Y = [X, Y] \tag{7.17}$$

Remark. Let's compare the Lie derivative and the covariant derivative. The former is defined on any manifold whereas the latter requires extra structure (a connection). Equation (7.17) reveals that the Lie derivative wrt X at p depends on X_p *and the first derivatives of* X at p. The covariant derivative wrt X at p depends only on X_p, which enables us to remove X to define the tensor ∇T, a covariant generalization of partial differentiation. It is not possible to define a corresponding tensor $\mathcal{L}T$ using the Lie derivative. Only $\mathcal{L}_X T$ makes sense.

Exercises (examples sheet 3).

1. Derive the formula for the Lie derivative of a covector ω in a coordinate basis:
$$(\mathcal{L}_X \omega)_\mu = X^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu X^\nu \tag{7.18}$$
Show that this can be written in the basis-independent form (where ∇ is the Levi-Civita connection)
$$(\mathcal{L}_X \omega)_a = X^b \nabla_b \omega_a + \omega_b \nabla_a X^b \tag{7.19}$$

2. Show that the Lie derivative of the metric in a coordinate basis is
$$(\mathcal{L}_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu X^\rho + g_{\rho\nu} \partial_\mu X^\rho \tag{7.20}$$
and that this can be written in the basis-independent form
$$(\mathcal{L}_X g)_{ab} = \nabla_a X_b + \nabla_b X_a \tag{7.21}$$

Remark. If ϕ_t is a symmetry transformation of T (for all t) then $\mathcal{L}_X T = 0$. If ϕ_t are a 1-parameter group of isometries then $\mathcal{L}_X g = 0$, i.e.,

$$\nabla_a X_b + \nabla_b X_a = 0 \tag{7.22}$$

This is *Killing's equation* and solutions are called *Killing vector fields*. Consider the case in which there exists a chart for which the metric tensor does not depend on some coordinate z. Then equation (7.20) reveals that $\partial/\partial z$ is a Killing vector field. Conversely, if the metric admits a Killing vector field then equation (7.13) demonstrates that one can introduce coordinates such that the metric tensor components are independent of one of the coordinates.

Lemma. Let X^a be a Killing vector field and let V^a be tangent to an affinely parameterized geodesic. Then $X_a V^a$ is constant along the geodesic.

Proof. The derivative of $X_a V^a$ along the geodesic is

$$\begin{aligned} \frac{d}{d\tau}(X_a V^a) = V(X_a V^a) &= \nabla_V (X_a V^a) = V^b \nabla_b (X_a V^a) \\ &= V^a V^b \nabla_b X_a + X_a V^b \nabla_b V^a \end{aligned} \tag{7.23}$$

The first term vanishes because Killing's equation implies that $\nabla_b X_a$ is antisymmetric. The second term vanishes by the geodesic equation.

Exercise. Let $J^a = T^a{}_b X^b$ where T_{ab} is the energy-momentum tensor and X^b is a Killing vector field. Show that $\nabla_a J^a = 0$, i.e., J^a is a *conserved current.*

Chapter 8

Linearized theory

8.1 The linearized Einstein equation

The nonlinearity of the Einstein equation makes it very hard to solve. However, in many circumstances, gravity is not strong and spacetime can be regarded as a perturbation of Minkowski spacetime. More precisely, we assume our spacetime manifold is $M = \mathbb{R}^4$ and that there exist globally defined "almost inertial" coordinates x^μ for which the metric can be written

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \tag{8.1}$$

with the weakness of the gravitational field corresponding to the components of $h_{\mu\nu}$ being small compared to 1. Note that we are dealing with a situation in which we have *two* metrics defined on spacetime, namely g_{ab} and the Minkowski metric η_{ab}. The former is supposed to be the physical metric, i.e., free particles move on geodesics of g_{ab}.

In linearized theory we regard $h_{\mu\nu}$ as the components of a tensor field in the sense of special relativity, i.e., it transforms as a tensor under Lorentz transformations of the coordinates x^μ.

To leading order in the perturbation, the inverse metric is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \tag{8.2}$$

where we define

$$h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma} \tag{8.3}$$

To prove this, just check that $g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho$ to linear order in the perturbation. Here, and henceforth, we shall raise and lower indices using the Minkowski metric $\eta_{\mu\nu}$. To leading order this agrees with raising and lowering with $g_{\mu\nu}$. We shall determine the Einstein equation to first order in the perturbation $h_{\mu\nu}$.

To first order, the Christoffel symbols are

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}\left(h_{\sigma\nu,\rho} + h_{\sigma\rho,\nu} - h_{\nu\rho,\sigma}\right), \tag{8.4}$$

the Riemann tensor is (neglecting $\Gamma\Gamma$ terms since they are second order in the perturbation):

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \eta_{\mu\tau}\left(\partial_\rho \Gamma^{\tau}_{\nu\sigma} - \partial_\sigma \Gamma^{\tau}_{\nu\rho}\right) \\ &= \frac{1}{2}\left(h_{\mu\sigma,\nu\rho} + h_{\nu\rho,\mu\sigma} - h_{\nu\sigma,\mu\rho} - h_{\mu\rho,\nu\sigma}\right) \end{aligned} \tag{8.5}$$

and the Ricci tensor is

$$R_{\mu\nu} = \partial^\rho \partial_{(\mu} h_{\nu)\rho} - \frac{1}{2}\partial^\rho\partial_\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h, \tag{8.6}$$

where ∂_μ denotes $\partial/\partial x^\mu$ as usual, and

$$h = h^{\mu}{}_{\mu} \tag{8.7}$$

To first order, the Einstein tensor is

$$G_{\mu\nu} = \partial^\rho \partial_{(\mu} h_{\nu)\rho} - \frac{1}{2}\partial^\rho\partial_\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h - \frac{1}{2}\eta_{\mu\nu}\left(\partial^\rho\partial^\sigma h_{\rho\sigma} - \partial^\rho\partial_\rho h\right). \tag{8.8}$$

The Einstein equation equates this to $8\pi T_{\mu\nu}$ (which must therefore be assumed to be small, otherwise spacetime would not be nearly flat). It is convenient to define

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}, \tag{8.9}$$

with inverse

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu}, \qquad (\bar{h} = \bar{h}^{\mu}{}_{\mu} = -h) \tag{8.10}$$

The linearized Einstein equation is then (exercise)

$$-\frac{1}{2}\partial^\rho\partial_\rho \bar{h}_{\mu\nu} + \partial^\rho\partial_{(\mu}\bar{h}_{\nu)\rho} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho\partial^\sigma \bar{h}_{\rho\sigma} = 8\pi T_{\mu\nu} \tag{8.11}$$

We must now discuss the gauge symmetry present in this theory. We argued above that diffeomorphisms are gauge transformations in GR. A manifold M with metric g and energy-momentum tensor T is physically equivalent to M with metric $\phi_*(g)$ and energy momentum tensor $\phi_*(T)$ if ϕ is a diffeomorphism. Now we are restricting attention to metrics of the form (8.1). Hence we must consider which diffeomorphisms preserve this form. A general diffeomorphism would lead to $(\phi_*(\eta))_{\mu\nu}$ very different from $\text{diag}(-1, 1, 1, 1)$ and hence such a diffeomorphism would not preserve (8.1). However, if we consider a 1-parameter family of diffeomorphisms ϕ_t then ϕ_0 is the identity map, so if t is small then ϕ_t is close to the identity and hence will have a small effect, i.e., $(\phi_{t*}(\eta))_{\mu\nu}$ will be close to diag$(-1,1,1,1)$ and the form (8.1) will be preserved. For small t, we can use the definition of the Lie derivative to deduce that, for any tensor T

$$\begin{aligned}(\phi_{-t})_*(T) &= T + t\mathcal{L}_X T + \mathcal{O}(t^2) \\ &= T + \mathcal{L}_\xi T + \mathcal{O}(t^2)\end{aligned} \tag{8.12}$$

where X^a is the vector field that generates ϕ_t and $\xi^a = tX^a$. Note that ξ^a is small so we treat it as a first order quantity. If we apply this result to the energy-momentum tensor, evaluating in our chart x^μ, then the first term is small (by assumption) so the second term is higher order and can be neglected. Hence the energy-momentum tensor is gauge-invariant to first order. The same is true for any tensor that vanishes in the unperturbed spacetime, e.g. the Riemann tensor. However, for the metric we have

$$(\phi_{-t})_*(g) = g + \mathcal{L}_\xi g + \ldots = \eta + h + \mathcal{L}_\xi \eta + \ldots \tag{8.13}$$

where we have neglected $\mathcal{L}_\xi h$ because this is a higher order quantity (as ξ and h both are small). Comparing this with (8.1) we deduce that h and $h + \mathcal{L}_\xi \eta$ describe physically equivalent metric perturbations. Therefore linearized GR has the gauge symmetry $h \to h + \mathcal{L}_\xi \eta$ for small ξ^μ. In our chart x^μ, we can use (7.21) to write $(\mathcal{L}_\xi \eta)_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ and so the gauge symmetry is

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{8.14}$$

There is a close analogy with electromagnetism in flat spacetime, where we can introduce an electromagnetic potential A_μ, a 4-vector obeying $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$. This has the gauge symmetry

$$A_\mu \to A_\mu + \partial_\mu \Lambda \tag{8.15}$$

for some scalar Λ. In this case, one can choose Λ to impose the *gauge condition* $\partial^\mu A_\mu = 0$. Similarly, in linearized GR we can choose ξ_μ to impose the gauge condition

$$\partial^\nu \bar{h}_{\mu\nu} = 0. \tag{8.16}$$

To see this, note that under the gauge transformation (8.14) we have

$$\partial^\nu \bar{h}_{\mu\nu} \to \partial^\nu \bar{h}_{\mu\nu} + \partial^\nu \partial_\nu \xi_\mu \tag{8.17}$$

so if we choose ξ_μ to satisfy the wave equation $\partial^\nu \partial_\nu \xi_\mu = -\partial^\nu \bar{h}_{\mu\nu}$ (which we can solve using a Green function) then we reach the gauge (8.16). This is called variously Lorenz, de Donder, or harmonic gauge. In this gauge, the linearized Einstein equation reduces to

$$\partial^\rho \partial_\rho \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{8.18}$$

Hence, in this gauge, each component of $\bar{h}_{\mu\nu}$ satisfies the wave equation with a source given by the energy-momentum tensor. Given appropriate boundary conditions, the solution can be determined using a Green function.

8.2 The Newtonian limit

We will now see how GR reduces to Newtonian theory in the limit of non-relativistic motion and a weak gravitational field. To do this, we could reintroduce factors of c and try to expand everything in powers of $1/c$ since we expect Newtonian theory to be valid as $c \to \infty$. We will follow an equivalent approach in which we stick to our convention $c = 1$ but introduce a small dimensionless parameter $0 < \epsilon \ll 1$ such that a factor of ϵ appears everywhere that a factor of $1/c$ would appear.

We assume that, for some choice of almost-inertial coordinates $x^\mu = (t, x^i)$, the 3-velocity of any particle $v^i = dx^i/d\tau$ is $\mathcal{O}(\epsilon)$. Recall that in Newtonian theory we had $v^2 \sim |\Phi|$ (Fig. 1.1) and so we expect the gravitational field to be $\mathcal{O}(\epsilon^2)$. We assume that

$$h_{00} = \mathcal{O}(\epsilon^2), \qquad h_{0i} = \mathcal{O}(\epsilon^3), \qquad h_{ij} = \mathcal{O}(\epsilon^2) \tag{8.19}$$

We'll see below how the additional factor of ϵ in h_{0i} emerges.

Since the matter which generates the gravitational field is assumed to move non-relativistically, time derivatives of the gravitational field will be small compared to spatial derivatives. Let L denote the length scale over which $h_{\mu\nu}$ varies, i.e., if X denotes some component of $h_{\mu\nu}$ then $|\partial_i X| = \mathcal{O}(X/L)$. Our assumption of small time derivatives is

$$\partial_0 X = \mathcal{O}(\epsilon X/L) \tag{8.20}$$

For example, in Newtonian theory, the gravitational field of a body of mass M at position $\mathbf{x}(t)$ is $\Phi = -M/|\mathbf{x} - \mathbf{x}(t)|$ which obeys these formulae with $L = |\mathbf{x} - \mathbf{x}(t)|$ and $|\dot{\mathbf{x}}| = \mathcal{O}(\epsilon)$.

Consider the equations for a timelike geodesic. The Lagrangian is (adding a hat to avoid confusion with the length L)

$$\hat{L} = (1 - h_{00})\dot{t}^2 - 2h_{0i}\dot{t}\dot{x}^i - (\delta_{ij} + h_{ij})\,\dot{x}^i\dot{x}^j \tag{8.21}$$

where a dot denotes a derivative with respect to proper time τ. Our non-relativistic assumption implies that $\dot{x}^i = \mathcal{O}(\epsilon)$. From the definition of proper time we have $\hat{L} = 1$ and hence

$$(1 - h_{00})\dot{t}^2 - \delta_{ij}\dot{x}^i\dot{x}^j = 1 + \mathcal{O}(\epsilon^4) \tag{8.22}$$

Rearranging gives

$$\dot{t} = 1 + \frac{1}{2}h_{00} + \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j + \mathcal{O}(\epsilon^4) \tag{8.23}$$

The Euler-Lagrange equation for x^i is

$$\begin{aligned} \frac{d}{d\tau}\left[-2h_{0i}\dot{t} - 2\left(\delta_{ij} + h_{ij}\right)\dot{x}^j\right] &= -h_{00,i}\dot{t}^2 - 2h_{0j,i}\dot{t}\dot{x}^j - h_{jk,i}\dot{x}^j\dot{x}^k \\ &= -h_{00,i} + \mathcal{O}(\epsilon^4/L) \end{aligned} \tag{8.24}$$

The LHS is $-2\ddot{x}^i$ plus subleading terms. Retaining only the leading order terms gives

$$\ddot{x}^i = \frac{1}{2}h_{00,i} \tag{8.25}$$

Finally we can convert τ derivatives on the LHS to t derivatives using the chain rule and (8.23) to obtain

$$\frac{d^2x^i}{dt^2} = -\partial_i\Phi \tag{8.26}$$

where

$$\Phi \equiv -\frac{1}{2}h_{00} \tag{8.27}$$

We have recovered the equation of motion for a test body moving in a Newtonian gravitational field Φ. Note that the weak equivalence principle automatically is satisfied: it follows from the fact that test bodies move on geodesics.

Exercise. Show that the corrections to (8.26) are $\mathcal{O}(\epsilon^4/L)$. You can argue as follows. (8.25) implies $\ddot{x}^i = \mathcal{O}(\epsilon^2/L)$. Now use (8.23) to show $\ddot{t} = \mathcal{O}(\epsilon^3/L)$. Expand out the derivative on the LHS of (8.24) to show that the corrections to (8.25) are $\mathcal{O}(\epsilon^4/L)$. Finally convert τ derivatives to t derivatives using (8.23).

The next thing we need to show is that Φ satisfies the Poisson equation (1.1). First consider the energy-momentum tensor. Assume that one can ascribe a 4-velocity u^a to the matter. Our non-relativistic assumption implies

$$u^i = \mathcal{O}(\epsilon), \qquad u^0 = 1 + \mathcal{O}(\epsilon^2) \tag{8.28}$$

where the second equality follows from $g_{ab}u^au^b = -1$. The energy-density in the rest-frame of the matter is

$$\rho \equiv u^a u^b T_{ab} \tag{8.29}$$

Recall that $-T_{0i}$ is the momentum density measured by an observer at rest in these coordinates so we expect $-T_{0i} \sim \rho u_i = \mathcal{O}(\rho\epsilon)$. T_{ij} will have a part $\sim \rho u_i u_j = \mathcal{O}(\rho\epsilon^2)$ arising from the motion of the matter. It will also have a contribution from the stresses in the matter. Under all but the most extreme circumstances, these are small compared to ρ. For example, in the rest frame of a perfect fluid, stresses are determined by the pressure p. The speed of sound in the fluid is C where $C^2 = dp/d\rho \sim p/\rho$. Our non-relativistic assumption is that $C = \mathcal{O}(\epsilon)$ hence $p = \mathcal{O}(\rho\epsilon^2)$. This is true in the Solar system, where $p/\rho \sim |\Phi| \sim 10^{-5}$ at the centre of the Sun. Hence we make the following assumptions

$$T_{00} = \rho(1 + \mathcal{O}(\epsilon^2)), \qquad T_{0i} = \mathcal{O}(\rho\epsilon), \qquad T_{ij} = \mathcal{O}(\rho\epsilon^2) \tag{8.30}$$

where the first equality follows from (8.29) and other two equalities.

Finally, we consider the linearized Einstein equation. Equation (8.19) implies that

$$\bar{h}_{00} = \mathcal{O}(\epsilon^2), \qquad \bar{h}_{0i} = \mathcal{O}(\epsilon^3), \qquad \bar{h}_{ij} = \mathcal{O}(\epsilon^2) \tag{8.31}$$

Using our assumption about time derivatives being small compared to spatial derivatives, equation (8.18) becomes

$$\nabla^2 \bar{h}_{00} = -16\pi\rho(1 + \mathcal{O}(\epsilon^2)), \qquad \nabla^2 \bar{h}_{0i} = \mathcal{O}(\rho\epsilon), \qquad \nabla^2 \bar{h}_{ij} = \mathcal{O}(\rho\epsilon^2) \tag{8.32}$$

If we impose boundary conditions that the metric perturbation (gravitational field) should decay at large distance then these equations can be solved using a Green function as in (1.2). The factors of ϵ on the RHS above imply that the resulting solutions satisfy

$$\bar{h}_{0i} = \mathcal{O}(\bar{h}_{00}\epsilon) = \mathcal{O}(\epsilon^3), \qquad \bar{h}_{ij} = \mathcal{O}(\bar{h}_{00}\epsilon^2) = \mathcal{O}(\epsilon^4) \tag{8.33}$$

Since $h_{0i} = \bar{h}_{0i}$, this explains why we had to assume $h_{0i} = \mathcal{O}(\epsilon^3)$. From the second result, we have $\bar{h}_{ii} = \mathcal{O}(\epsilon^4)$ and hence $\bar{h} = -\bar{h}_{00} + \mathcal{O}(\epsilon^4)$. We can use (8.10) to recover $h_{\mu\nu}$. This gives $h_{00} = (1/2)\bar{h}_{00} + \mathcal{O}(\epsilon^4)$ and so, using (8.27) we obtain Newton's law of gravitation:

$$\nabla^2 \Phi = 4\pi\rho(1 + \mathcal{O}(\epsilon^2)) \tag{8.34}$$

We also obtain $h_{ij} = (1/2)\bar{h}_{00}\delta_{ij} + \mathcal{O}(\epsilon^4)$ and so

$$h_{ij} = -2\Phi\delta_{ij} + \mathcal{O}(\epsilon^4) \tag{8.35}$$

This justifies the metric used in (1.16). The expansion of various quantities in powers of ϵ can be extended to higher orders. As we have seen, the Newtonian approximation requires only the $\mathcal{O}(\epsilon^2)$ term in h_{00}. The next order, *post-Newtonian*, approximation corresponds to including $\mathcal{O}(\epsilon^4)$ terms in h_{00}, $\mathcal{O}(\epsilon^3)$ terms in h_{0i} and $\mathcal{O}(\epsilon^2)$ in h_{ij}. Equation (8.35) gives h_{ij} to $\mathcal{O}(\epsilon^2)$. The above analysis also lets us write the $\mathcal{O}(\epsilon^3)$ term in h_{0i} in terms of T_{0i} and a Green function. However, to obtain the $\mathcal{O}(\epsilon^4)$ term in h_{00} one has to go beyond linearized theory.

8.3 Gravitational waves

In vacuum, the linearized Einstein equation reduces to the source-free wave equation:

$$\partial^\rho \partial_\rho \bar{h}_{\mu\nu} = 0 \tag{8.36}$$

so the theory admits gravitational wave solutions. As usual for the wave equation, we can build a general solution as a superposition of plane wave solutions. So let's look for a plane wave solution:

$$\bar{h}_{\mu\nu}(x) = \mathrm{Re}\left(H_{\mu\nu} e^{ik_\rho x^\rho}\right) \tag{8.37}$$

where $H_{\mu\nu}$ is a constant symmetric complex matrix describing the polarization of the wave and k^μ is the (real) wavevector. We shall suppress the Re in all subsequent equations. The wave equation reduces to

$$k_\mu k^\mu = 0 \tag{8.38}$$

so the wavevector k^μ must be null hence these waves propagate at the speed of light relative to the background Minkowski metric. The gauge condition (8.16) gives

$$k^\nu H_{\mu\nu} = 0, \tag{8.39}$$

i.e. the waves are transverse.

Example. Consider the null vector $k^\mu = \omega(1, 0, 0, 1)$. Then $\exp(ik_\mu x^\mu) = \exp(-i\omega(t - z))$ so this describes a wave of frequency ω propagating at the speed of light in the z-direction. The transverse condition reduces to

$$H_{\mu 0} + H_{\mu 3} = 0. \tag{8.40}$$

Returning to the general case, the condition (8.16) does not eliminate all gauge freedom. Consider a gauge transformation (8.14). From equation (8.17), we see that this preserves the gauge condition (8.16) if ξ_μ obeys the wave equation:

$$\partial^\nu \partial_\nu \xi_\mu = 0. \tag{8.41}$$

Hence there is a residual gauge freedom which we can exploit to simplify the solution. Take

$$\xi_\mu(x) = X_\mu e^{ik_\rho x^\rho} \tag{8.42}$$

which satisfies (8.41) because k_μ is null. Using

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial^\rho \xi_\rho \tag{8.43}$$

we see that the residual gauge freedom in our case is

$$H_{\mu\nu} \to H_{\mu\nu} + i\left(k_\mu X_\nu + k_\nu X_\mu - \eta_{\mu\nu} k^\rho X_\rho\right) \tag{8.44}$$

Exercise. Show that the residual gauge freedom can be used to achieve "longitudinal gauge":

$$H_{0\mu} = 0 \tag{8.45}$$

but this still does not determine X_μ uniquely, and the freedom remains to impose the additional "trace-free" condition

$$H^\mu{}_\mu = 0. \tag{8.46}$$

In this gauge, we have

$$h_{\mu\nu} = \bar{h}_{\mu\nu}. \tag{8.47}$$

Example. Return to our wave travelling in the z-direction. The longitudinal gauge condition combined with the transversality condition (8.40) gives $H_{3\mu} = 0$. Using the trace-free condition now gives

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{8.48}$$

where the solution is specified by the two constants H_+ and $H_\times$ corresponding to two independent polarizations. So gravitational waves are transverse and have two possible polarizations. This is one way of interpreting the statement that the gravitational field has two degrees of freedom per spacetime point.

How would one detect a gravitational wave? An observer could set up a family of test particles locally. The displacement vector S^a from the observer to any particle is governed by the geodesic deviation equation. (We are taking S^a to be infinitesimal, i.e., what we called $\delta s S^a$ previously.) So we can use this equation to predict what the observer would see. We have to be careful here. It would be natural to write out the geodesic deviation equation using the almost inertial coordinates and therby determine S^μ. But how would we relate this to observations? S^μ are the components of S^a with respect to a certain basis, so how would we determine whether the variation in S^μ arises from variation of S^a or from variation of the basis? With a bit more thought, one can make this approach work but we shall take a different approach.

Consider an observer following a geodesic in a general spacetime. Our observer will be equipped with a set of measuring rods with which to measure distances.

At some point p on his worldline we could introduce a local inertial frame with spatial coordinates X, Y, Z in which the observer is at rest. Imagine that the observer sets up measuring rods of unit length pointing in the X, Y, Z directions at p. Mathematically, this defines an orthonormal basis $\{e_\alpha\}$ for $T_p(M)$ (we use α to label the basis vectors because we are using μ for our almost inertial coordinates) where $e_0^a = u^a$ (his 4-velocity) and e_i^a $(i = 1, 2, 3)$ are spacelike vectors satisfying

$$u_a e_i^a = 0, \qquad g_{ab} e_i^a e_j^b = \delta_{ij} \tag{8.49}$$

In Minkowski spacetime, this basis can be extended to the observer's entire worldline by taking the basis vectors to have constant components (in an inertial frame), i.e., they do not depend on proper time τ. In particular, this implies that the orthonormal basis is *non-rotating*. Since the basis vectors have constant components, they are parallelly transported along the worldline. Hence, in curved spacetime, the analogue of this is to take the basis vectors to be parallelly transported along the worldline. For u^a, this is automatic (the worldline is a geodesic). But for e_i it gives

$$u^b \nabla_b e_i^a = 0 \tag{8.50}$$

As we discussed previously, if the e_i^a are specified at any point p then this equation determines them uniquely along the whole worldline. Furthermore, the basis remains orthonormal because parallel transport preserves inner products (examples sheet 2). The basis just constructed is sometimes called a *parallelly transported frame*. It is the kind of basis that would be constructed by an observer freely falling with 3 gyroscopes whose spin axes define the spatial basis vectors. Using such a basis we can be sure that an increase in a component of S^a is really an increase in the distance to the particle in a particular direction, rather than a basis-dependent effect.

Now imagine this observer sets up a family of test particles near his worldline. The deviation vector to any infinitesimally nearby particle satisfies the geodesic deviation equation

$$u^b \nabla_b (u^c \nabla_c S_a) = R_{abcd} u^b u^c S^d \tag{8.51}$$

Contract with e_α^a and use the fact that the basis is parallelly transported to obtain

$$u^b \nabla_b [u^c \nabla_c (e_\alpha^a S_a)] = R_{abcd} e_\alpha^a u^b u^c S^d \tag{8.52}$$

Now $e_\alpha^a S_a$ is a *scalar* hence the equation reduces to

$$\frac{d^2 S_\alpha}{d\tau^2} = R_{abcd} e_\alpha^a u^b u^c e_\beta^d S^\beta \tag{8.53}$$

where τ is the observer's proper time and $S_\alpha = e_\alpha^a S_a$ is one of the components of S_a in the parallelly transported frame. On the RHS we've used $S^d = e_\beta^d S^\beta$.

So far, the discussion has been general but now let's specialize to our gravitational plane wave. On the RHS, R_{abcd} is a first order quantity so we only need to evaluate the other quantities to leading order, i.e., we can evaluate them as if spacetime were flat. Assume that the observer is at rest in the almost inertial coordinates. To leading order, $u^\mu = (1,0,0,0)$ hence

$$\frac{d^2 S_\alpha}{d\tau^2} \approx R_{\mu 00\nu} e^\mu_\alpha e^\nu_\beta S^\beta \tag{8.54}$$

Using the formula for the perturbed Riemann tensor (8.5) and $h_{0\mu} = 0$ we obtain

$$\frac{d^2 S_\alpha}{d\tau^2} \approx \frac{1}{2}\frac{\partial^2 h_{\mu\nu}}{\partial t^2} e^\mu_\alpha e^\nu_\beta S^\beta \tag{8.55}$$

In Minkowski spacetime we could take e_i^a aligned with the x, y, z axes respectively, i.e., $e_1^\mu = (0,1,0,0)$, $e_2^\mu = (0,0,1,0)$ and $e_3^\mu = (0,0,0,1)$. We can use the same results here because we only need to evaluate e^μ_α to leading order. Using $h_{0\mu} = h_{3\mu} = 0$ we then have

$$\frac{d^2 S_0}{d\tau^2} = \frac{d^2 S_3}{d\tau^2} = 0 \tag{8.56}$$

to this order of approximation. Hence the observer sees no relative acceleration of the test particles in the z-direction, i.e, the direction of propagation of the wave. Let the observer set up initial conditions so that S_0 and its first derivatives vanish at $\tau = 0$. Then S_0 will vanish for all time. If the derivative of S_3 vanishes initially then S_3 will be constant. The same is not true for the other components.

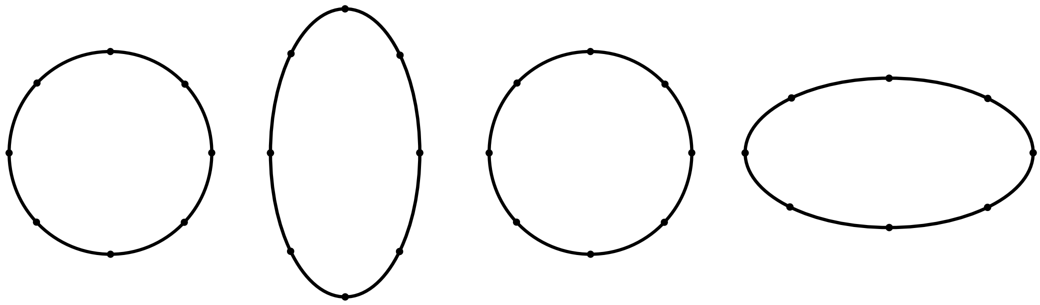
We can choose our almost inertial coordinates so that the observer has coordinates $x^\mu \approx (\tau, 0, 0, 0)$ (i.e. $t = \tau$ to leading order along the observer's worldline). For a + polarized wave we then have

$$\frac{d^2 S_1}{d\tau^2} = -\frac{1}{2}\omega^2 |H_+| \cos(\omega\tau - \alpha) S_1, \qquad \frac{d^2 S_2}{d\tau^2} = \frac{1}{2}\omega^2 |H_+| \cos(\omega\tau - \alpha) S_2 \tag{8.57}$$

where we have replaced t by τ in $\partial^2 h_{\mu\nu}/\partial t^2$ and $\alpha = \arg H_+$. Since H_+ is small we can solve this perturbatively: the leading order solution is $S_1 = \bar{S}_1$, a constant (assuming that we set up initial condition so that the particles are at rest to leading order). Similarly $S_2 = \bar{S}_2$. Now we can plug these leading order solutions into the RHS of the above equations and integrate to determine the solution up to first order (again choosing constants of integration so that the particles would be at rest in the absense of the wave)

$$S_1(\tau) \approx \bar{S}_1 \left(1 + \frac{1}{2}|H_+| \cos(\omega\tau - \alpha)\right), \quad S_2(\tau) \approx \bar{S}_2 \left(1 - \frac{1}{2}|H_+| \cos(\omega\tau - \alpha)\right) \tag{8.58}$$

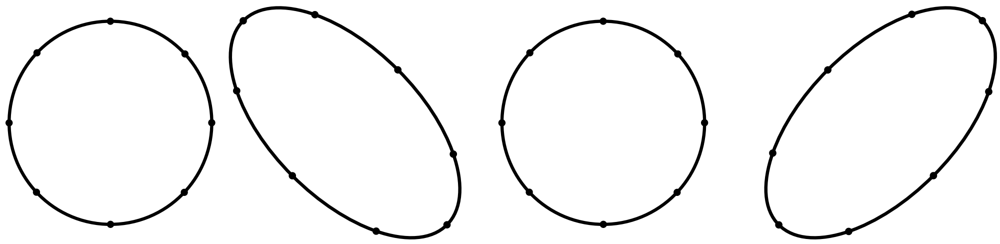
This reveals that particles are displaced outwards in the x-direction whilst being displaced inwards in the y-direction, and vice-versa. $\bar{S}_1$ and $\bar{S}_2$ give the average displacement. If the particles are arranged in the xy plane with $\bar{S}_1^2 + \bar{S}_2^2 = R^2$ then they form a circle of radius R when $\omega\tau = \alpha + \pi/2$. This will be deformed into an ellipse, then back to a circle, then an ellipse again (Fig 8.1).



$\omega\tau = \alpha + \frac{1}{2}\pi$ $\quad \omega\tau = \alpha + \pi$ $\quad \omega\tau = \alpha + \frac{3}{2}\pi$ $\quad \omega\tau = \alpha + 2\pi$

Figure 8.1: Geodesic deviation caused by + polarized wave.

Exercise. Show that the corresponding result for a $\times$ polarized wave is the same, just rotated through 45° (Fig. 8.2).



$\omega\tau = \alpha + \frac{1}{2}\pi$ $\quad \omega\tau = \alpha + \pi$ $\quad \omega\tau = \alpha + \frac{3}{2}\pi$ $\quad \omega\tau = \alpha + 2\pi$

Figure 8.2: Geodesic deviation caused by $\times$ polarized wave.

Gravitational wave detectors look for the changes in position of test masses caused by the above effect. For example, the two LIGO observatories (in the US, see Fig. 8.3) each have two perpendicular tunnels, each 4 km long. There are test masses (analogous to the particles above) at the end of each arm (tunnel) and where the arms meet. A beam splitter is attached to the test mass where the arms meet. A laser signal is split and sent down each arm, where it reflects off mirrors attached to the test masses at the ends of the arms. The signals are recombined and interferometry used to detect whether there has been any change in the length

Figure 8.3: The LIGO Hanford observatory in Washington state, USA. There is another LIGO observatory in Louisiana. (Image credit: LIGO.)

difference of the two arms. The effect that is being looked for is tiny: plausible sources of gravitational waves give $H_+, H_\times \sim 10^{-21}$ (see below) so the relative length change of each arm is $\delta L/L \sim 10^{-21}$. The resulting δL is a tiny fraction of the wavelength of the laser light but the resulting tiny phase difference between the two laser signals is detectable!

Gravitational wave detectors have been operating for several decades, gradually improving in sensitivity. The first direct detection of gravitational waves was announced by the LIGO collaboration in February 2016. As we will explain below, there is also very good *indirect* evidence for the existence of gravitational waves. We will discuss all of this in more detail later.

8.4 The field far from a source

Let's return to the linearized Einstein equation with matter:

$$\partial^\rho \partial_\rho \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{8.59}$$

Since each component of $\bar{h}_{\mu\nu}$ satisfies the inhomogeneous wave equation, the solution can be solved using the same retarded Green function that one uses in electromagnetism:

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = 4 \int d^3x' \frac{T_{\mu\nu}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tag{8.60}$$

where $|\mathbf{x}-\mathbf{x}'|$ is calculated using the Euclidean metric.

Assume that the matter is confined to a compact region near the origin of size d (e.g. let d be the radius of the smallest sphere that encloses the matter). Then, far from the source we have $r \equiv |\mathbf{x}| \gg |\mathbf{x}'| \sim d$ so we can expand

$$|\mathbf{x}-\mathbf{x}'|^2 = r^2 - 2\mathbf{x}\cdot\mathbf{x}' + \mathbf{x}'^2 = r^2\left(1-(2/r)\hat{\mathbf{x}}\cdot\mathbf{x}' + \mathcal{O}(d^2/r^2)\right) \tag{8.61}$$

(where $\hat{\mathbf{x}} = \mathbf{x}/r$) hence

$$|\mathbf{x}-\mathbf{x}'| = r - \hat{\mathbf{x}}\cdot\mathbf{x}' + \mathcal{O}(d^2/r) \tag{8.62}$$

$$T_{\mu\nu}(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}') = T_{\mu\nu}(t',\mathbf{x}') + \hat{\mathbf{x}}\cdot\mathbf{x}'(\partial_0 T_{\mu\nu})(t',\mathbf{x}') + \ldots \tag{8.63}$$

where

$$t' = t - r \tag{8.64}$$

Now let τ denote the time scale on which $T_{\mu\nu}$ is varying so $\partial_0 T_{\mu\nu} \sim T_{\mu\nu}/\tau$. For example, if the source is a binary star system, then τ is the orbital period. The second term in (8.63) is of order $(d/\tau)T_{\mu\nu}$. Note that d is the time it takes light to cross the region containing the matter. Hence $d/\tau \ll 1$ will be satisfied if the matter is moving non-relativistically. We assume this henceforth. This implies that the second term in (8.63) is negligible compared to the first and so

$$\bar{h}_{ij}(t,\mathbf{x}) \approx \frac{4}{r}\int d^3x'\, T_{ij}(t',\mathbf{x}') \qquad t' = t - r \tag{8.65}$$

Here we are considering just the spatial components of $\bar{h}_{\mu\nu}$, i.e., $\bar{h}_{ij}$. Other components can be obtained from the gauge condition (8.16), which gives

$$\partial_0 \bar{h}_{0i} = \partial_j \bar{h}_{ji}, \qquad \partial_0 \bar{h}_{00} = \partial_i \bar{h}_{0i} \tag{8.66}$$

Given $\bar{h}_{ij}$, the first equation can be integrated to determine $\bar{h}_{0i}$ and the second can then be integrated to determine $\bar{h}_{00}$.

The integral in (8.65) can be evaluated as follows. Since the matter is compactly supported, we can freely integrate by parts and discard surface terms (note also that t' does not depend on $\mathbf{x}'$). We can also use energy-momentum conservation, which to this order is just $\partial_\nu T^{\mu\nu} = 0$. Let's drop the primes on the coordinates in the integral for now.

$$\begin{aligned}
\int d^3x\, T^{ij} &= \int d^3x\, \left[\partial_k(T^{ik}x^j) - (\partial_k T^{ik})x^j\right] \\
&= -\int d^3x\, (\partial_k T^{ik})x^j \qquad \text{drop surface term} \\
&= \int d^3x (\partial_0 T^{i0})x^j \qquad \text{conservation law} \\
&= \partial_0 \int d^3x\, T^{0i}x^j
\end{aligned} \tag{8.67}$$

We can now symmetrize this equation on ij to get

$$\begin{aligned}
\int d^3x\, T^{ij} &= \partial_0 \int d^3x\, T^{0(i}x^{j)} \\
&= \partial_0 \int d^3x \left[\frac{1}{2}\partial_k\left(T^{0k}x^i x^j\right) - \frac{1}{2}(\partial_k T^{0k})x^i x^j\right] \\
&= -\frac{1}{2}\partial_0 \int d^3x\, (\partial_k T^{0k})x^i x^j \qquad \text{integration by parts} \\
&= \frac{1}{2}\partial_0 \int d^3x\, (\partial_0 T^{00})x^i x^j \qquad \text{conservation law} \\
&= \frac{1}{2}\partial_0\partial_0 \int d^3x\, T^{00}x^i x^j \\
&= \frac{1}{2}\ddot{I}_{ij}(t) \qquad (8.68)
\end{aligned}$$

where

$$I_{ij}(t) = \int d^3x T_{00}(t,\mathbf{x})x^i x^j \qquad (8.69)$$

(Note that $T_{00} = T^{00}$ and $T_{ij} = T^{ij}$ to leading order.) Hence we have

$$\bar{h}_{ij}(t,\mathbf{x}) \approx \frac{2}{r}\ddot{I}_{ij}(t-r) \qquad (8.70)$$

This result is valid when $r \gg d$ and $\tau \gg d$.

I_{ij} is the second moment of the energy density. It is a tensor in the Cartesian sense, i.e., it transforms in the usual way under rotations of the coordinates x^i. (The zeroth moment is the total energy in matter $\int d^3x T_{00}$, the first moment is the energy dipole $\int d^3x T_{00}x^i$.)

The result (8.70) describes a disturbance propagating outwards from the source at the speed of light. If the source exhibits oscillatory motion (e.g. a binary star system) then $\bar{h}_{ij}$ will describe waves with the same period τ as the motion of the source.

The first equation in (8.66) gives

$$\partial_0\bar{h}_{0i} \approx \partial_j\left(\frac{2}{r}\ddot{I}_{ij}(t-r)\right) \qquad (8.71)$$

so integrating with respect to time gives (using $\partial_i r = x_i/r \equiv \hat{x}_i$)

$$\begin{aligned}
\bar{h}_{0i} \approx \partial_j\left(\frac{2}{r}\dot{I}_{ij}(t-r)\right) &= -\frac{2\hat{x}_j}{r^2}\dot{I}_{ij}(t-r) - \frac{2\hat{x}_j}{r}\ddot{I}_{ij}(t-r) \\
&\approx -\frac{2\hat{x}_j}{r}\ddot{I}_{ij}(t-r) \qquad (8.72)
\end{aligned}$$

In the final line we have assumed that we are in the *radiation zone* $r \gg \tau$. This allows us to neglect the term proportional to $\dot{I}$ because it is smaller than the term we have retained by a factor τ/r. In the radiation zone, space and time derivatives are of the same order of magnitude. (Note that, even for a non-relativistic source, the Newtonian approximation breaks down in the radiation zone because (8.20) is violated.)

Similarly integrating the second equation in (8.66) gives

$$\bar{h}_{00} \approx \partial_i \left(-\frac{2\hat{x}_j}{r} \dot{I}_{ij}(t-r) \right) \approx \frac{2\hat{x}_i \hat{x}_j}{r} \ddot{I}_{ij}(t-r) \tag{8.73}$$

These expressions are not quite right because when we integrated (8.66) we should have included an arbitrary time-independent term in $\bar{h}_{0i}$, leading to a term in $\bar{h}_{00}$ linear in time, as well as an arbitrary time-independent term in $\bar{h}_{00}$. The latter term can be determined by returning to (8.60) which to leading order gives

$$\bar{h}_{00} \approx \frac{4E}{r} \tag{8.74}$$

where E is the total energy of the matter

$$E = \int d^3\mathbf{x}' T_{00}(t', \mathbf{x}') \tag{8.75}$$

Why have we not rediscovered the leading-order time-dependent piece (8.73)? This piece is smaller than the time-independent part by a factor of d^2/τ^2 so we'd have to go to higher order in the expansion of (8.60) to find this term (and also the term linear in t, which appears at higher order). Note that energy-momentum conservation gives

$$\partial_0 E = \int d^3\mathbf{x}' (\partial_0 T_{00})(t', \mathbf{x}') = \int d^3\mathbf{x}' (\partial_i T_{i0})(t', \mathbf{x}') = 0 \tag{8.76}$$

and hence E is a constant. Similarly,

$$\bar{h}_{0i} \approx -\frac{4P_i}{r} \tag{8.77}$$

where P_i is the total 3-momentum

$$P_i = -\int d^3\mathbf{x}' T_{0i}(t', \mathbf{x}') \tag{8.78}$$

This is also constant by energy-momentum conservation. Note that the term in $\bar{h}_{00}$ that is linear in t is proportional to P_i.

Remark. We will show below that gravitational waves carry away energy (they also carry away momentum). So why is the total energy of matter constant? In fact, the total energy of matter is *not* constant but one has to go beyond linearized theory to see this: one would have to correct the equation for energy-momentum conservation to take account of the perturbation to the connection. But then we would have to correct the LHS of the linearized Einstein equation, including second order terms for consistency with the new conservation law of the RHS. So to see this effect requires going beyond linearized theory.

A final simplification is possible: we are free to choose our almost inertial coordinates to correspond to the "centre of momentum frame", i.e., $P_i = 0$. If we do this then E is just the total mass of the matter, which we shall denote M. Putting everything together we have

$$\bar{h}_{00}(t, \mathbf{x}) \approx \frac{4M}{r} + \frac{2\hat{x}_i\hat{x}_j}{r}\ddot{I}_{ij}(t-r), \qquad \bar{h}_{0i}(t, \mathbf{x}) \approx -\frac{2\hat{x}_j}{r}\ddot{I}_{ij}(t-r) \tag{8.79}$$

To recap, we have assumed $r \gg \tau \gg d$ and work in the centre of momentum frame. In $\bar{h}_{00}$ we have retained the second term, even though it is subleading relative to the first, because this is the leading order time-dependent term.

8.5 The energy in gravitational waves

We see that the gravitational waves arise when I_{ij} varies in time. Gravitational waves carry energy away from the souce. Calculating this is subtle: as discussed previously, there is no local energy density for the gravitational field. To explain the calculation we must go to second order in perturbation theory. At second order, our metric $\eta_{\mu\nu} + h_{\mu\nu}$ will not satisfy the Einstein equation so we have to add a second order correction, writing

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + h^{(2)}_{\mu\nu} \tag{8.80}$$

The idea is that if the components of $h_{\mu\nu}$ are $\mathcal{O}(\epsilon)$ then the components of $h^{(2)}_{\mu\nu}$ are $\mathcal{O}(\epsilon^2)$.

Now we calculate the Einstein tensor to second order. The first order term is what we calculated before (equation (8.8)). We shall call this $G^{(1)}_{\mu\nu}[h]$. The second order terms are either linear in $h^{(2)}$ or quadratic in h. The terms linear in $h^{(2)}$ can be calculated by setting h to zero. This is exactly the same calculation we did before but with h replaced by $h^{(2)}$. Hence the result will be (8.8) with $h \to h^{(2)}$, which we denote $G^{(1)}_{\mu\nu}[h^{(2)}]$. Therefore to second order we have

$$G_{\mu\nu}[g] = G^{(1)}_{\mu\nu}[h] + G^{(1)}_{\mu\nu}[h^{(2)}] + G^{(2)}_{\mu\nu}[h] \tag{8.81}$$

where $G^{(2)}_{\mu\nu}[h]$ is the part of $G_{\mu\nu}$ that is quadratic in h. This is:

$$G^{(2)}_{\mu\nu}[h] = R^{(2)}_{\mu\nu}[h] - \frac{1}{2}R^{(1)}[h]h_{\mu\nu} - \frac{1}{2}R^{(2)}[h]\eta_{\mu\nu} \tag{8.82}$$

where $R^{(2)}_{\mu\nu}[h]$ is the term in the Ricci tensor that is quadratic in h. $R^{(1)}$ and $R^{(2)}$ are the terms in the Ricci scalar which are linear and quadratic in h respectively. The latter can be written

$$R^{(2)}[h] = \eta^{\mu\nu}R^{(2)}_{\mu\nu}[h] - h^{\mu\nu}R^{(1)}_{\mu\nu}[h] \tag{8.83}$$

Exercise (examples sheet 3). Show that

$$\begin{aligned} R^{(2)}_{\mu\nu}[h] &= \frac{1}{2}h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} - h^{\rho\sigma}\partial_\rho\partial_{(\mu}h_{\nu)\sigma} + \frac{1}{4}\partial_\mu h_{\rho\sigma}\partial_\nu h^{\rho\sigma} + \partial^\sigma h^\rho{}_\nu\partial_{[\sigma}h_{\rho]\mu} \\ &+ \frac{1}{2}\partial_\sigma\left(h^{\sigma\rho}\partial_\rho h_{\mu\nu}\right) - \frac{1}{4}\partial^\rho h\partial_\rho h_{\mu\nu} - \left(\partial_\sigma h^{\rho\sigma} - \frac{1}{2}\partial^\rho h\right)\partial_{(\mu}h_{\nu)\rho} \end{aligned} \tag{8.84}$$

For simplicity, assume that no matter is present. At first order, the linearized Einstein equation is $G^{(1)}_{\mu\nu}[h] = 0$ as before. At second order we have

$$G^{(1)}_{\mu\nu}[h^{(2)}] = 8\pi t_{\mu\nu}[h] \tag{8.85}$$

where

$$t_{\mu\nu}[h] \equiv -\frac{1}{8\pi}G^{(2)}_{\mu\nu}[h] \tag{8.86}$$

Equation (8.85) is the equation of motion for $h^{(2)}$. If h satisfies the linear Einstein equation then we have $R^{(1)}_{\mu\nu}[h] = 0$ so the above results give

$$t_{\mu\nu}[h] = -\frac{1}{8\pi}\left(R^{(2)}_{\mu\nu}[h] - \frac{1}{2}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}[h]\eta_{\mu\nu}\right) \tag{8.87}$$

Consider now the contracted Bianchi identity $g^{\mu\rho}\nabla_\rho G_{\mu\nu} = 0$. Expanding this, at first order we get

$$\partial^\mu G^{(1)}_{\mu\nu}[h] = 0 \tag{8.88}$$

for *arbitrary* first order perturbation h (i.e. not assuming that h satisfies any equation of motion). At second order we get

$$\partial^\mu\left(G^{(1)}_{\mu\nu}[h^{(2)}] + G^{(2)}_{\mu\nu}[h]\right) + hG^{(1)}[h] = 0 \tag{8.89}$$

where the final term denotes schematically the terms that arise from the first order change in the inverse metric and the Christoffel symbols in $g^{\mu\rho}\nabla_\rho$. Now, since (8.88) holds for arbitrary h, it holds if we replace h with $h^{(2)}$ so $\partial^\mu G^{(1)}_{\mu\nu}[h^{(2)}] = 0$.

If we now assume that h satisfies its equation of motion $G^{(1)}[h] = 0$ then the final term in (8.89) vanishes and this equation reduces to

$$\partial^\mu t_{\mu\nu} = 0. \tag{8.90}$$

Hence $t_{\mu\nu}$ is a symmetric tensor (in the sense of special relativity) that is (i) quadratic in the linear perturbation h, (ii) *conserved* if h satisfies its equation of motion, and (iii) appears on the RHS of the second order Einstein equation (8.85). This is a natural candidate for the energy momentum tensor of the linearized gravitational field.

Unfortunately, $t_{\mu\nu}$ suffers from a major problem: it is not invariant under a gauge transformation (8.14). This is how the impossibility of localizing gravitational energy arises in linearized theory.

Nevertheless, it can be shown that the *integral* of t_{00} over a surface of constant time $t = x^0$ *is* gauge invariant provided one considers $h_{\mu\nu}$ that decays at infinity, and restricts to gauge transformations which preserve this property. This integral provides a satisfactory notion of the *total* energy in the linearized gravitational field. Hence gravitational energy does exist, but it cannot be localized.

One can use the second order Einstein equation (8.85) to convert the integral defining the energy, which is quadratic in $h_{\mu\nu}$, into a surface integral at infinity which is linear in $h^{(2)}_{\mu\nu}$. In fact the latter can be made fully nonlinear: these surface integrals make sense in any spacetime which is "asymptotically flat", irrespective of whether or not the linearized approximation holds in the interior. This notion of energy is referred to as the *ADM energy*. This is constant in time but there is a related quantity called the *Bondi energy*, a non-increasing function of time. The rate of change of this can be interpreted as the rate of energy loss in gravitational waves.

We shall follow a less rigorous, but more intuitive, approach in which we convert $t_{\mu\nu}$ into a gauge-invariant quantity by *averaging*. For any point p, consider some region R of $\mathbb{R}^4$ of typical coordinate size a (in all directions) centred on p. Define the average of a tensor $X_{\mu\nu}$ at p by

$$\langle X_{\mu\nu} \rangle = \int_R X_{\mu\nu}(x) W(x) d^4x \tag{8.91}$$

where the averaging function $W(x)$ is positive, satisfies $\int_R W d^4x = 1$, and tends smoothly to zero on ∂R. Note that it makes sense to integrate $X_{\mu\nu}$ because we are treating it as a tensor in Minkowski spacetime, so we can add tensors at different points.

We are interested in averaging in the region far from the source, in which we have gravitational radiation with some typical wavelength λ (in the notation used above $\lambda \sim \tau$). Assume that the components of $X_{\mu\nu}$ have typical size x. Since the wavelength of the radiation is λ, $\partial_\mu X_{\nu\rho}$ will have components of typical size x/λ. But the average is

$$\langle \partial_\mu X_{\nu\rho} \rangle = -\int_R X_{\nu\rho}(x) \partial_\mu W(x) d^4x \tag{8.92}$$

where we have integrated by parts and used $W = 0$ on ∂R. Now $\partial_\mu W$ has components of order W/a so the RHS above has components of order x/a. Hence if we choose $a \gg \lambda$ then averaging has the effect of reducing $\partial_\mu X_{\nu\rho}$ by a factor of $\lambda/a \ll 1$. So if we choose $a \gg \lambda$ then we can neglect total derivatives inside averages. This implies that we are free to integrate by parts inside averages:

$$\langle A \partial B \rangle = \langle \partial(AB) \rangle - \langle (\partial A) B \rangle \approx -\langle (\partial A) B \rangle \tag{8.93}$$

because $\langle \partial(AB) \rangle$ is a factor λ/a smaller than $\langle B \partial A \rangle$. Henceforth we assume $a \gg \lambda$ so we can exploit these properties.

Exercises (examples sheet 3).

1. Use the linearized Einstein equation to show that, in vacuum,

$$\langle \eta^{\mu\nu} R^{(2)}_{\mu\nu}[h] \rangle = 0 \tag{8.94}$$

   Hence the second term in $t_{\mu\nu}[h]$ averages to zero.

2. Show that

$$\langle t_{\mu\nu} \rangle = \frac{1}{32\pi} \langle \partial_\mu \bar{h}_{\rho\sigma} \partial_\nu \bar{h}^{\rho\sigma} - \frac{1}{2} \partial_\mu \bar{h} \partial_\nu \bar{h} - 2 \partial_\sigma \bar{h}^{\rho\sigma} \partial_{(\mu} \bar{h}_{\nu)\sigma} \rangle \tag{8.95}$$

3. Show that $\langle t_{\mu\nu} \rangle$ is gauge invariant.

Hence we *can* obtain a gauge invariant energy momentum tensor as long as we average over a region much larger than the wavelength of the the gravitational radiation we are studying. This might be a rather large region: the LIGO detector looks for waves with frequency around 100Hz, corresponding to a wavelength $\lambda \sim$ 3000km.

Exercise. Calculate $\langle t_{\mu\nu} \rangle$ for the plane gravitational wave solution discussed above. Show that

$$\langle t_{\mu\nu} \rangle = \frac{1}{32\pi} \left(|H_+|^2 + |H_\times|^2 \right) k_\mu k_\nu = \frac{\omega^2}{32\pi} \left(|H_+|^2 + |H_\times|^2 \right) \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \tag{8.96}$$

As one would expect, there is a constant flux of energy and momentum travelling at the speed of light in the z-direction.

8.6 The quadrupole formula

Now we are ready to calculate the energy loss from a compact source due to gravitational radiation. The averaged energy flux 3-vector is $-\langle t_{0i}\rangle$. Consider a large sphere r = constant far outside the source. The unit normal to the sphere (in a surface of constant t) is $\hat{x}_i$. Hence the average total energy flux across this sphere, i.e., the average power radiated across the sphere is

$$\langle P\rangle = -\int r^2 d\Omega \langle t_{0i}\rangle \hat{x}_i \tag{8.97}$$

where $d\Omega$ is the usual volume element on a unit S^2.

Calculating this is just a matter of substituting the results of section 8.4 into (8.95). Since these results apply in harmonic gauge, we have

$$\begin{aligned}\langle t_{0i}\rangle &= \frac{1}{32\pi}\langle \partial_0 \bar{h}_{\rho\sigma}\partial_i \bar{h}^{\rho\sigma} - \frac{1}{2}\partial_0 \bar{h}\partial_i \bar{h}\rangle \\ &= \frac{1}{32\pi}\langle \partial_0 \bar{h}_{jk}\partial_i \bar{h}_{jk} - 2\partial_0 \bar{h}_{0j}\partial_i \bar{h}_{0j} + \partial_0 \bar{h}_{00}\partial_i \bar{h}_{00} - \frac{1}{2}\partial_0 \bar{h}\partial_i \bar{h}\rangle \end{aligned} \tag{8.98}$$

Since $\bar{h}_{jk}(t,\mathbf{x}) = (2/r)\ddot{I}_{jk}(t-r)$ we have

$$\partial_0 \bar{h}_{jk} = \frac{2}{r}\dddot{I}_{jk}(t-r) \tag{8.99}$$

and

$$\partial_i \bar{h}_{jk} = \left(-\frac{2}{r}\dddot{I}_{jk}(t-r) - \frac{2}{r^2}\ddot{I}_{jk}(t-r)\right)\hat{x}_i \tag{8.100}$$

The second term is smaller than the first by a factor $\tau/r \ll 1$ and so negligible for large enough r. Hence

$$-\frac{1}{32\pi}\int r^2 d\Omega \langle \partial_0 \bar{h}_{jk}\partial_i \bar{h}_{jk}\rangle \hat{x}_i = \frac{1}{2}\langle \dddot{I}_{ij}\dddot{I}_{ij}\rangle_{t-r} \tag{8.101}$$

On the RHS, the average is a time average, taken over an interval $a \gg \lambda \sim \tau$ centered on the retarded time $t-r$.

Next we have $\bar{h}_{0j} = (-2\hat{x}_k/r)\ddot{I}_{jk}(t-r)$ hence

$$\partial_0 \bar{h}_{0j} = -\frac{2\hat{x}_k}{r}\dddot{I}_{jk}(t-r) \qquad \partial_i \bar{h}_{0j} \approx \frac{2\hat{x}_k}{r}\dddot{I}_{jk}(t-r)\hat{x}_i \tag{8.102}$$

where in the second expressions we have used $\tau/r \ll 1$ to neglect the terms arising from differentiation of $\hat{x}_k/r$. Hence

$$-\frac{1}{32\pi}\int r^2 d\Omega \langle -2\partial_0 \bar{h}_{0j}\partial_i \bar{h}_{0j}\rangle \hat{x}_i = -\frac{1}{4\pi}\langle \dddot{I}_{jk}\dddot{I}_{jl}\rangle_{t-r}\int d\Omega \hat{x}_k \hat{x}_l \tag{8.103}$$

Now recall the following from Cartesian tensors: $\int d\Omega \hat{x}_k \hat{x}_l$ is isotropic (rotationally invariant) and hence must equal $\lambda \delta_{kl}$ for some constant λ. Taking the trace fixes $\lambda = 4\pi/3$. Hence the RHS above is

$$-\frac{1}{3}\langle \dddot{I}_{ij} \dddot{I}_{ij} \rangle_{t-r} \tag{8.104}$$

Next we use $\bar{h}_{00} = 4M/r + (2\hat{x}_j \hat{x}_k / r)\ddot{I}_{jk}(t-r)$ to give

$$\partial_0 \bar{h}_{00} = \frac{2\hat{x}_j \hat{x}_k}{r} \dddot{I}_{jk}(t-r) \tag{8.105}$$

and

$$\partial_i \bar{h}_{00} \approx \left(-\frac{4M}{r^2} - \frac{2\hat{x}_j \hat{x}_k}{r} \dddot{I}_{jk}(t-r) \right) \hat{x}_i \approx -\frac{2\hat{x}_j \hat{x}_k}{r} \dddot{I}_{jk}(t-r) \hat{x}_i \tag{8.106}$$

where we've neglected terms arising from differentiation of $\hat{x}_j \hat{x}_k / r$ in the first equality because in the radiation zone ($\tau / r \ll 1$) they're negligible compared to the second term we've retained. In the second equality we've neglected the first term in brackets because this leads to a term in the integral proportional to $\langle \dddot{I}_{jk} \rangle$, which is the average of a derivative and therefore negligible. Hence we have

$$-\frac{1}{32\pi} \int r^2 d\Omega \langle \partial_0 \bar{h}_{00} \partial_i \bar{h}_{00} \rangle \hat{x}_i = \frac{1}{8\pi} \langle \dddot{I}_{ij} \dddot{I}_{kl} \rangle_{t-r} X_{ijkl} \tag{8.107}$$

where

$$X_{ijkl} = \int d\Omega \hat{x}_i \hat{x}_j \hat{x}_k \hat{x}_l \tag{8.108}$$

is another isotropic integral which we'll evaluate below.

Next we use $\bar{h} = \bar{h}_{jj} - \bar{h}_{00}$ and the above results to obtain

$$\partial_0 \bar{h} = \frac{2}{r} \dddot{I}_{jj}(t-r) - \frac{2\hat{x}_j \hat{x}_k}{r} \dddot{I}_{jk}(t-r) \tag{8.109}$$

$$\partial_i \bar{h} = \left(-\frac{2}{r} \dddot{I}_{jj}(t-r) + \frac{2\hat{x}_j \hat{x}_k}{r} \dddot{I}_{jk}(t-r) \right) \hat{x}_i \tag{8.110}$$

and hence (using the result above for $\int d\Omega \hat{x}_i \hat{x}_j$)

$$-\frac{1}{32\pi} \int r^2 d\Omega \langle -\frac{1}{2} \partial_0 \bar{h} \partial_i \bar{h} \rangle \hat{x}_i = \langle -\frac{1}{4} \dddot{I}_{jj} \dddot{I}_{kk} + \frac{1}{6} \dddot{I}_{jj} \dddot{I}_{kk} - \frac{1}{16\pi} \dddot{I}_{ij} \dddot{I}_{kl} X_{ijkl} \rangle \tag{8.111}$$

Putting everything together we have

$$\langle P \rangle_t = \langle \frac{1}{6} \dddot{I}_{ij} \dddot{I}_{ij} - \frac{1}{12} \dddot{I}_{ii} \dddot{I}_{jj} + \frac{1}{16\pi} \dddot{I}_{ij} \dddot{I}_{kl} X_{ijkl} \rangle_{t-r} \tag{8.112}$$

To evaluate X_{ijkl}, we use the fact that any isotropic Cartesian tensor is a product of δ factors and ϵ factors. In the present case, X_{ijkl} has rank 4 so we can only use δ terms so we must have $X_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$ for some α, β, γ. The symmetry of X_{ijkl} implies that $\alpha = \beta = \gamma$. Taking the trace on ij and on kl indices then fixes $\alpha = 4\pi/15$. The final term above is therefore

$$\frac{1}{60}\langle \dddot{I}_{ii}\dddot{I}_{jj} + 2\, \dddot{I}_{ij}\dddot{I}_{ij}\rangle \tag{8.113}$$

and hence

$$\langle P\rangle_t = \frac{1}{5}\langle \dddot{I}_{ij}\dddot{I}_{ij} - \frac{1}{3}\, \dddot{I}_{ii}\dddot{I}_{jj}\rangle_{t-r} \tag{8.114}$$

Finally we define the energy quadrupole tensor, which is the traceless part of I_{ij}

$$Q_{ij} = I_{ij} - \frac{1}{3}I_{kk}\delta_{ij} \tag{8.115}$$

We then have

$$\langle P\rangle_t = \frac{1}{5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle_{t-r} \tag{8.116}$$

This is the *quadrupole formula* for energy loss via gravitational wave emission. It is valid in the radiation zone far from a non-relativistic source, i.e., for $r \gg \tau \gg d$.

We conclude that a body whose quadrupole tensor is varying in time will emit gravitational radiation. A spherically symmetric body has $Q_{ij} = 0$ and so will not radiate, in agreement with *Birkhoff's theorem*, which asserts that the unique spherically symmetric solution of the vacuum Einstein equation is the Schwarzschild solution. Hence the spacetime outside a spherically symmetric body is time independent because it is described by the Schwarzschild solution.

8.7 Comparison with electromagnetic radiation

In electromagnetic theory, given a charge distribution ρ we can define the total charge

$$Q = \int d^3x\rho \tag{8.117}$$

and the electric dipole moment

$$\mathbf{D} = \int d^3x\rho\mathbf{x} \tag{8.118}$$

Similarly for a matter distribution with energy density T_{00} we have defined the total energy

$$E = \int d^3xT_{00} \tag{8.119}$$

and we can define the centre of mass

$$\mathbf{X}(t) = E^{-1} \int d^3x T_{00}(t, \mathbf{x})\mathbf{x} \tag{8.120}$$

Electromagnetic radiation is produced by the motion of charge. Of course charge is conserved so Q does not vary with time, just like E does not vary with time. Hence there is no monpole radiation in either electromagnetism or gravity. However, $\mathbf{D}$ can change with time, and changing $\mathbf{D}$ leads to emission of electromagnetic radiation with power

$$\langle P \rangle_t = \frac{1}{12\pi\epsilon_0} \langle |\ddot{\mathbf{D}}|^2 \rangle_{t-r} \tag{8.121}$$

Since the analogue of $\mathbf{D}$ is $\mathbf{X}$, one might expect gravitational dipole radiation when $\mathbf{X}$ varies. However, we have

$$E\ddot{\mathbf{X}} = \dot{\mathbf{P}} = 0 \tag{8.122}$$

where $\mathbf{P}$ is the total momentum of the mass distribution, which is conserved. Hence there is no gravitational analogue of electric dipole radiation: it is forbidden by linear momentum conservation.

In electromagnetic theory, a varying *magnetic* dipole moment also produces radiation, although this is usually much weaker than electric dipole radiation. The magnetic dipole is

$$\mu = \int d^3x \, \mathbf{x} \times \mathbf{j} \tag{8.123}$$

where $\mathbf{j}$ is the electric current. The analogue of a magnetic dipole moment for a mass distribution is

$$\mathbf{J} = \int d^3x \, \mathbf{x} \times (\rho \mathbf{u}) \tag{8.124}$$

where $\mathbf{u}$ is the local velocity of matter (i.e. $T_{0i} \approx -\rho u_i$ as in section 8.2). But this is simply the total angular momentum of the matter, which is again conserved. So there is no gravitational analogue of magnetic dipole radiation: it is forbidden by conservation of angular momentum.

These arguments "explain" why there is no monopole or dipole gravitational radiation. Gravitational quadrupole radiation is analogous to electric quadrupole radiation in electromagnetic theory, which is the leading order effect when the electric and magnetic dipoles do not vary.

It is easy to detect electromagnetic radiation but gravitational radiation is very hard to detect. This is because gravitational waves interact only very weakly with matter (or with each other). This is equivalent to the familiar statement that gravity is very weak force - the weakest known force in Nature, and much weaker than the electromagnetic force.

One way to see this is to consider the energy flux $\mathcal{F}$ of a plane gravitational wave. For example, take a wave with $h \sim 10^{-21}$ and $\omega \sim 100\text{Hz}$, the kind of signal the LIGO detectors search for. From the 03 component of (8.96) we have, reinstating factors of G and c to give quantity with the correct dimensions

$$\mathcal{F} \sim \frac{\omega^2 c}{32\pi G} h^2 \sim 0.01\text{Wm}^{-2} \tag{8.125}$$

where we are just working to an order of magnitude. This is the same as the energy flux around 30m from a 100W light bulb. Of course an electromagnetic flux of this magnitude is easily detectable - your eyes are doing it now. (However, the light has much higher frequency so maybe a better comparison is with 100Hz electromagnetic waves, i.e., low frequency radio waves, and these would also be easy to detect at a flux of 0.01Wm^{-2}.) But to detect the same energy flux in gravitational waves requires spending over a billion dollars to construct a state of the art detector! A large energy flux produces only a very small effect on the detector because gravity interacts with matter so weakly.

The weakness of gravity has some advantanges. Because gravitational waves do not interact much with matter, they do not suffer much distortion as they propagate through the Universe. Unlike electromagnetic waves, they are not absorbed or scattered by matter. So the waves received by a detector are essentially the same as the waves emitted by the source, adjusted for cosmological expansion.

8.8 Gravitational waves from binary systems

A fairly common astrophysical system with a time-varying quadrupole is a binary system, consisting of a pair of stars orbiting their common centre of mass. Consider the case when the stars both have mass M, their separation is d and the orbital period is τ so the angular velocity is $\omega \sim \tau^{-1}$. Then Newton's second law gives $M\omega^2 d \sim M^2/d^2$ which gives $\omega \sim M^{1/2}d^{-3/2}$. The quadrupole tensor has components of typical size Md^2 so the third derivative is of size $Md^2/\tau^3 \sim Md^2\omega^3 \sim (M/d)^{5/2}$. Hence we obtain the order of magnitude estimate

$$P \sim (M/d)^5. \tag{8.126}$$

The power radiated in gravitational waves is greatest when M/d is as large as possible. Note the large power (5) on the RHS of this equation: if M/d decreases by an order of magnitude then P decreases by 5 orders of magnitude. So P is a rapidly decreasing function of M/d. To get a large P we need the system to have M/d as small as possible, i.e., it has to be as compact as possible.

To understand the size of P, recall that we've used units $G = c = 1$ in (8.126). Reinstating units gives

$$P \sim (M/d)^5 L_{\text{Planck}} \tag{8.127}$$

where $L_{\rm Planck}$ is the *Planck luminosity*

$$L_{\rm Planck} = \frac{c^5}{G} \approx 4 \times 10^{52} \text{ W} \tag{8.128}$$

This is a mind-bogglingly enormous luminosity. The electromagnetic luminosity of the Sun is $L_\odot \approx 4 \times 10^{26}$W $\approx 10^{-26} L_{\rm Planck}$. There are roughly 10^{10} galaxies in the observable Universe, so if a typical galaxy contains 10^{11} stars we can estimate the electromagnetic luminosity of all the stars in the Universe as $10^{21} L_\odot \approx 10^{-5} L_{\rm Planck}$. Hence a binary with $M/d \gtrsim 10^{-1}$ would emit more power in gravitational radiation than all the stars in the Universe emit in electromagnetic radiation!

How big can M/d be? Obviously d cannot be smaller than the size R of the stars themselves. However, a normal star has radius $R \gg M$. For example, the Sun has $R \approx 7 \times 10^5$km and $M \approx 1.5$km so $M/R \sim 2 \times 10^{-6}$ hence a binary made of Sun-like stars would have $M/d \ll 10^{-6}$ as $d \gg R$. To obtain a larger amount of radiation we need to consider binary systems made of much more compact bodies, i.e., bodies with M/R as large as possible. The most compact objects in Nature are *black holes*, whose size is given by the *Schwarzschild radius* $R = 2M$ (anything more compact than this would collapse to form a black hole: see the Black Holes course). Almost as compact are *neutron stars*: stars made of nuclear matter held together by gravity, like a giant atomic nucleus. So the binaries which are expected to emit the most gravitational radiation are tightly bound NS/NS, NS/BH or BH/BH systems (NS: neutron star, BH: black hole).

The emission of gravitational radiation causes the shape of the orbit to change over time. To a good approximation, valid when the stars are far apart and moving non-relativistically, we can calculate this by letting the radius of the Newtonian orbit vary slowly with time. The energy of a Newtonian orbit is $E \sim -M^2/d$ so d decreases as the system loses energy via gravitational radiation. Hence the orbital period $\tau \sim d^{3/2} M^{-1/2}$ also decreases. To calculate how d varies with time, we equate dE/dt with $-P$. (See Examples sheet 4.) This approximation can be improved by including higher order, post-Newtonian, effects.

This prediction of GR has been confirmed observationally. In 1974, Hulse and Taylor detected a *binary pulsar*. This is a NS/NS binary in which one of the stars is a pulsar, i.e., it emits a beam of radio waves in a certain direction. This star is rotating very rapidly and the beam (which is not aligned with the rotation axis) periodically points in our direction. Hence we receive pulses of radiation from the star. The period between successive pulses (about 0.05s) is very stable and has been measured to very high accuracy. Therefore it acts like a clock that we can observe from Earth. Using this clock we can determine the orbital period (about 7.75h) of the binary system, again with good accuracy. The emission of gravitational waves causes the period of the orbit to decrease by about 10μs per year. This small effect has been measured and the result confirms the quadrupole formula to an accuracy of 0.3% (the accuracy increases the longer the system is observed). This is very strong *indirect* evidence for the existence of gravitational waves, for which Hulse and Taylor received the Nobel Prize in 1993. (The gravitational wave luminosity of the Hulse-Taylor system is about $0.02L_\odot$.)

As a compact binary system loses energy to radiation, the radius of the orbit shrinks and eventually the two bodies in the system will collide and merge to form a single black hole (it is unlikely to be a neutron star because a NS cannot have a mass greater than about $2M_\odot$). As the system approaches merger, the velocities of the two bodies become very large, a significant fraction of the speed of light. For such a system, the post-Newtonian expansion is useless and the only way of predicting what will happen is to solve the Einstein equation numerically on a supercomputer. As the bodies approach merger, the luminosity P still increases in rough agreement with (8.126). Hence the strongest sources of gravitational waves are expected to be compact binaries just before merger.

To discuss the *direct* detection of gravitational waves from a merging compact binary, we need to estimate the amplitude of the gravitational waves from such a source. At a distance r, the above estimates give

$$\bar{h}_{ij} \sim \frac{Md^2}{\tau^2 r} \sim \frac{M^2}{dr} \tag{8.129}$$

We can use this to estimate the amplitude of waves arriving at a detector on Earth. We take r to be the distance within which we expect there to exist sufficiently many suitable binary system that at least one will merge within a reasonable time, say 1 year (we don't want to have to wait for 100 years to detect anything!). The process of gradual inspiral to final merger is very slow, taking billions of years for plausible initial conditions (see Examples sheet 4). This implies that r must be of cosmological size: of the order of 3×10^8 light years. Taking M to be about ten times the mass of the Sun and d to be ten times the Schwarzschild radius gives $h \sim 10^{-21}$ and waves with a frequency of $100 - 1000$ Hz. This is the kind of signal that the two LIGO detectors (in the US) and the VIRGO detector (in Italy) are designed to detect.

On 11 February 2016, the LIGO collaboration announced that it had made the first direct detection of gravitational waves on 14 September 2015 (Fig. 8.4). By comparing with the detailed predictions of General Relativity (determined using the post-Newtonian expansion and numerical simulations), it was deduced that these waves were emitted in the merger of a compact binary at a distance of 1.4×10^9 light years. The masses of the objects in the binary were estimated to be $36M_\odot$ and $29M_\odot$. Since these are well above the upper mass limit for a neutron star, it was deduced that this was a BH/BH binary. The merger produced a final BH with mass $62M_\odot$. The missing mass $3M_\odot$ was emitted as gravitational radiation. The detected signal lasted for about 0.5s. The gravitational wave luminosity at the merger was greater than the electromagnetic luminosity of all the stars in the visible Universe!

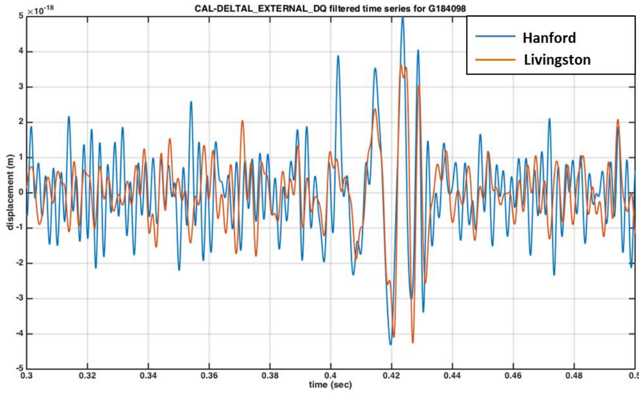


Figure 8.4: The signals detected by the two LIGO observatories on 14 September 2015. The vertical axis is the change in length of the arms of the detectors. (Image credit: LIGO.)

Gravitational waves from a second BH/BH merger were detected on 26 December 2015, this time with masses $14M_\odot$ and $8M_\odot$ merging to form a BH of mass $21M_\odot$, at a similar distance to the first detection. Based on these detections, it is expected that the LIGO detectors will detect roughly one BH/BH merger per month at their current sensitivity. Upgrades to the sensitivity are planned, which could lead to 1 detection per day by 2019!

Chapter 9

Differential forms

9.1 Introduction

Definition. Let M be a differentiable manifold. A p-form on M is an antisymmetric $(0, p)$ tensor field on M.

Remark. A 0-form is a function, a 1-form is a covector field.

Definition. The *wedge product* of a p-form X and a q-form Y is the $(p+q)$-form $X \wedge Y$ defined by

$$(X \wedge Y)_{a_1 \ldots a_p b_1 \ldots b_q} = \frac{(p+q)!}{p!q!} X_{[a_1 \ldots a_p} Y_{b_1 \ldots b_q]} \tag{9.1}$$

Exercise. Show that

$$X \wedge Y = (-1)^{pq} Y \wedge X \tag{9.2}$$

(so $X \wedge X = 0$ if p is odd); and

$$(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z) \tag{9.3}$$

i.e. the wedge product is associative so we don't need to include the brackets.

Remark. If we have a dual basis $\{f^\mu\}$ then the set of p-forms of the form $f^{\mu_1} \wedge f^{\mu_2} \ldots \wedge f^{\mu_p}$ give a basis for the space of all p-forms on M because we can write

$$X = \frac{1}{p!} X_{\mu_1 \ldots \mu_p} f^{\mu_1} \wedge f^{\mu_2} \ldots \wedge f^{\mu_p} \tag{9.4}$$

Definition. The *exterior derivative* of a p-form X is the $(p+1)$-form dX defined in a coordinate basis by

$$(dX)_{\mu_1 \ldots \mu_{p+1}} = (p+1) \partial_{[\mu_1} X_{\mu_2 \ldots \mu_{p+1}]} \tag{9.5}$$

Exercise. Show that this is independent of the choice of coordinate basis.

Remark. This reduces to our previous definition of d acting on functions when $p = 0$.

Lemma. If ∇ is a (torsion-free) connection then

$$(dX)_{a_1 \dots a_{p+1}} = (p+1)\nabla_{[a_1} X_{a_2 \dots a_{p+1}]} \tag{9.6}$$

Proof. The components of the LHS and RHS are equal in normal coordinates at r for any point r.

Exercises (examples sheet 4). Show that the exterior derivative enjoys the following properties:

$$d(dX) = 0 \tag{9.7}$$

$$d(X \wedge Y) = (dX) \wedge Y + (-1)^p X \wedge dY \tag{9.8}$$

(where Y is a q-form) and

$$d(\phi^* X) = \phi^* dX \tag{9.9}$$

(where $\phi : N \to M$), i.e. the exterior derivative commutes with pull-back.

Remark. The last property implies that the exterior derivative commutes with a Lie derivative:

$$\mathcal{L}_V(dX) = d(\mathcal{L}_V X) \tag{9.10}$$

where V is a vector field.

Definition. X is *closed* if $dX = 0$ everywhere. X is *exact* if there exists a $(p-1)$-form Y such that $X = dY$ everywhere.

Remark. Exact implies closed. The converse is true only locally:

Poincaré Lemma. If X is a closed p-form ($p \geq 1$) then for any $r \in M$ there exists a neighbourhood $\mathcal{O}$ of r and a $(p-1)$-form Y such that $X = dY$ in $\mathcal{O}$.

9.2 Connection 1-forms

Remark. Often in GR it is convenient to work with an orthonormal basis of vector fields $\{e^a_\mu\}$ obeying

$$g_{ab} e^a_\mu e^b_\nu = \eta_{\mu\nu} \tag{9.11}$$

where $\eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1)$. (In 4d, such a basis is sometimes called a *tetrad*.) (On a Riemannian manifold we do the same with $\eta_{\mu\nu}$ replaced by $\delta_{\mu\nu}$.) Since $\eta_{\mu\nu}$ is the metric in our basis, Greek tensor indices can be lowered with $\eta_{\mu\nu}$ and raised with $\eta^{\mu\nu}$.

Exercise. Show that the dual basis $\{f_a^\mu\}$ is given by

$$f_a^\mu = \eta^{\mu\nu}(e_\nu)_a \equiv e_a^\mu \qquad \Rightarrow \qquad e_a^\mu e_\nu^a = \delta_\nu^\mu \tag{9.12}$$

Remark. Here we have *defined* what it means to raise the Greek index labeling the basis vector. Henceforth any Greek index can be raised/lowered with the Minkowski metric.

Remark. We saw earlier (section 3.2) that any two orthonormal bases at are related by a Lorentz transformation which acts on the indices μ, ν:

$$e'^a_\mu = \left(A^{-1}\right)^\nu{}_\mu e^a_\nu, \qquad \eta_{\mu\nu} A^\mu{}_\rho A^\nu{}_\sigma = \eta_{\rho\sigma} \tag{9.13}$$

There is an important difference with special relativity: the Lorentz transformation $A^\mu{}_\nu$ need not be constant, it can depend on position. Working with orthonormal bases brings the equations of GR to a form in which Lorentz transformations are a *local* symmetry.

Lemma.

$$\eta_{\mu\nu} e_a^\mu e_b^\nu = g_{ab}, \qquad e_a^\mu e_\mu^b = \delta_a^b \tag{9.14}$$

Proof. Contract the LHS of the first equation with a basis vector e_ρ^b:

$$\eta_{\mu\nu} e_a^\mu e_b^\nu e_\rho^b = \eta_{\mu\nu} e_a^\mu \delta_\rho^\nu = \eta_{\mu\rho} e_a^\mu = (e_\rho)_a = g_{ab} e_\rho^b \tag{9.15}$$

Hence the first equation is true when contracted with any basis vector so it is true in general. This equation can be written as $e_a^\mu (e_\mu)_b = g_{ab}$. Raise the b index to get the second equation.

Definition. The *connection 1-forms* $\omega^\mu{}_\nu$ are (using the Levi-Civita connection)

$$\left(\omega^\mu{}_\nu\right)_a = e_b^\mu \nabla_a e_\nu^b \tag{9.16}$$

Exercise. Show that

$$\left(\omega^\mu{}_\nu\right)_a = \Gamma^\mu_{\nu\rho} e_a^\rho \tag{9.17}$$

where $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols.

Remark. The indices μ, ν on $\omega^\mu{}_\nu$ are *not* tensor indices: they do *not* transform correctly under Lorentz transformations. This is just the fact that the components of the connection are not tensor components. However, for each pair (μ, ν), we do have a well-defined 1-form $\omega^\mu{}_\nu$.

Lemma. $(\omega_{\mu\nu})_a = -(\omega_{\nu\mu})_a$.
Proof.

$$(\omega_{\mu\nu})_a = (e_\mu)_b \nabla_a e_\nu^b = \nabla_a \left((e_\mu)_b e_\nu^b\right) - e_\nu^b \nabla_a (e_\mu)_b = \nabla_a \eta_{\mu\nu} - (\omega_{\nu\mu})_a \tag{9.18}$$

and $\nabla_a \eta_{\mu\nu} = 0$ because $\eta_{\mu\nu}$ are just constant scalars.

Lemma. Regard e^μ_a as a 1-form. Then

$$de^\mu = -\omega^\mu{}_\nu \wedge e^\nu \tag{9.19}$$

Proof. From the definition of $\omega^\mu{}_\nu$ we have

$$\nabla_a e^b_\nu = (\omega^\mu{}_\nu)_a \, e^b_\mu \tag{9.20}$$

hence

$$\nabla_a (e_\mu)_b = (\omega_{\nu\mu})_a \, e^\nu_b = -\, (\omega_{\mu\nu})_a \, e^\nu_b \tag{9.21}$$

and so

$$(de^\mu)_{ab} = 2\nabla_{[a} e^\mu_{b]} = -2\, (\omega^\mu{}_\nu)_{[a} \, e^\nu_{b]} = -\, (\omega^\mu{}_\nu \wedge e^\nu)_{ab} \tag{9.22}$$

Remark. Evaluating (9.19) in our basis gives (cf (9.4))

$$de^\mu = -(\omega^\mu{}_\nu)_\rho e^\rho \wedge e^\nu = (\omega^\mu{}_{[\nu})_{\rho]} e^\nu \wedge e^\rho \tag{9.23}$$

and hence

$$(de^\mu)_{\nu\rho} = 2(\omega^\mu{}_{[\nu})_{\rho]}. \tag{9.24}$$

So if we calculated de^μ then we can read off $(\omega^\mu{}_{[\nu})_{\rho]}$. If we do this for all μ then the connection 1-forms can be read off because the antisymmetry $\omega_{\mu\nu} = -\omega_{\nu\mu}$ implies $(\omega_{\mu\nu})_\rho = (\omega_{\mu[\nu})_{\rho]} + (\omega_{\nu[\rho})_{\mu]} - (\omega_{\rho[\mu})_{\nu]}$. Since calculating de^μ is often quite easy, this provides a convenient method of calculating the connection 1-forms. In simple cases, one can read off $\omega^\mu{}_\nu$ by inspection. It is always a good idea to check the result by substituting back into (9.19).

Example. The Schwarzschild spacetime admits the obvious tetrad

$$e^0 = f dt, \qquad e^1 = f^{-1} dr, \qquad e^2 = r d\theta, \qquad e^3 = r \sin\theta d\phi \tag{9.25}$$

where $f = \sqrt{1 - 2M/r}$. We then have

$$de^0 = df \wedge dt + f d(dt) = f' dr \wedge dt = f' e^1 \wedge e^0 \tag{9.26}$$

$$de^1 = -f^{-2} f' dr \wedge dr = 0 \tag{9.27}$$

$$de^2 = dr \wedge d\theta = (f/r) e^1 \wedge e^2 \tag{9.28}$$

$$de^3 = \sin\theta dr \wedge d\phi + r\cos\theta d\theta \wedge d\phi = (f/r) e^1 \wedge e^3 + (1/r)\cot\theta e^2 \wedge e^3 \tag{9.29}$$

The first of these suggests we try $\omega^0{}_1 = f' e^0$. This would give $\omega_{01} = -f' e^0$ and hence $\omega_{10} = f' e^0$ so $\omega^1{}_0 = f' e^0$. This would make a vanishing contribution to de^1 ($\omega^1{}_0 \wedge e^0 = 0$), which looks promising. The third equation suggests we try $\omega^2{}_1 = (f/r)e^2$, which gives $\omega^1{}_2 = -(f/r)e^2$ which again would make a vanishing contribution to de^1. The final equation suggests $\omega^3{}_1 = (f/r)e^3$ and $\omega^3{}_2 = (1/r)\cot\theta\, e^3$ and again these will not spoil the second and third equations. So these connection 1-forms (and those related by $\omega_{\mu\nu} = -\omega_{\nu\mu}$) must be the correct answer.

Remark. From (4.8), the components of the covariant derivative of a vector field Y^a are

$$\nabla_\nu Y^\mu = e_\nu(Y^\mu) + \Gamma^\mu_{\rho\nu} V^\rho = \partial_\nu V^\mu + \omega^\mu{}_{\rho\nu} Y^\rho \tag{9.30}$$

where $\partial_\mu \equiv e^\alpha_\mu \partial_\alpha$, α refers to a coordinate basis, and $\omega^\mu{}_{\rho\nu} \equiv (\omega^\mu{}_\rho)_\nu$.

Exercise. What is the corresponding result for a $(1,1)$ tensor field?

9.3 Spinors in curved spacetime

Remark. We've seen how to define tensors in curved spacetime. But what about spinor fields, i.e., fields of non-integer spin? If we use orthonormal bases, this is straightforward because the structure of special relativity is manifest locally.

Remark. For now, we work at a single point p. Consider an infinitesimal Lorentz transformation at p

$$A^\mu{}_\nu = \delta^\mu{}_\nu + \alpha^\mu{}_\nu \tag{9.31}$$

for infinitesimal α. The condition that this be a Lorentz transformation (the second equation in (9.13) gives the restriction

$$\alpha_{\mu\nu} = -\alpha_{\nu\mu} \tag{9.32}$$

So an infinitesimal Lorentz transformation at p is described by an antisymmetric matrix. We now consider a representation of the Lorentz group at p in which the Lorentz transformation A is described by a matrix $D(A)$.

Definition. The *generators* of the Lorentz group in the representation D are matrices $T^{\mu\nu} = -T^{\nu\mu}$ defined by

$$D(A) = 1 + \frac{1}{2}\alpha_{\mu\nu} T^{\mu\nu} \tag{9.33}$$

when A is given by (9.31).

Example. Lorentz transformations were defined by looking at transformations of vectors. Let's work out the generators in this defining, *vector representation.* Under a Lorentz transformation, the components of a vector transform as $X'^\mu = A^\mu{}_\nu X^\nu = X^\mu + \alpha^\mu{}_\nu X^\nu$ so we must have

$$\frac{1}{2}\alpha_{\rho\sigma}\left(T^{\rho\sigma}\right)^\mu{}_\nu X^\nu = \alpha^\mu{}_\nu X^\nu = \alpha_{\rho\sigma}\eta^{\mu\rho}\delta^\sigma_\nu X^\nu \tag{9.34}$$

and hence, remembering the antisymmetry of α, the Lorentz generators in the vector representation, which we denote $M^{\mu\nu}$, have components

$$(M^{\rho\sigma})^\mu{}_\nu = \eta^{\mu\rho}\delta^\sigma_\nu - \eta^{\mu\sigma}\delta^\rho_\nu \tag{9.35}$$

From these we deduce the Lorentz algebra (the square brackets denote a matrix commutator)

$$[M^{\mu\nu}, M^{\rho\sigma}] = \ldots \tag{9.36}$$

Remark. A *finite* Lorentz transformation can be obtained by exponentiating:

$$A = \exp\left(\frac{1}{2}\alpha_{\mu\nu}M^{\mu\nu}\right) \tag{9.37}$$

We then have

$$D(A) = \exp\left(\frac{1}{2}\alpha_{\mu\nu}T^{\mu\nu}\right) \tag{9.38}$$

In these expressions, $\alpha_{\mu\nu} = -\alpha_{\mu\nu}$ are finite parameters describing the transformation.

Definition. The Dirac *gamma matrices* are a set of square matrices $\{\Gamma^\mu\}$ which obey

$$\Gamma^\mu\Gamma^\nu + \Gamma^\nu\Gamma^\mu = 2\eta^{\mu\nu} \tag{9.39}$$

Remark. In 4d spacetime, the smallest representation of the gamma matrices is given by the *Dirac representation* in which Γ^μ are 4×4 matrices. It is unique up to equivalence of representations.

Lemma. The matrices

$$T^{\mu\nu} = -\frac{1}{4}\left[\Gamma^\mu, \Gamma^\nu\right] \tag{9.40}$$

form a representation of the Lorentz algebra (9.36). This is the *Dirac representation* describing a particle of spin 1/2.

Remark. So far, we've worked at a single point p. Let's now allow p to vary. The following definition looks more elegant if one uses the language of vector bundles.

Definition. A *field in the Lorentz representation* D is a smooth map which, for any point p and any orthonormal basis $\{e_\mu\}$ defined in a neighbourhood of p, gives a vector ψ in the carrier space of the representation D, with the property that if $(p, \{e_\mu\})$ maps to ψ then $(p, \{e'_\mu\})$ maps to $D(A)\psi$ where $\{e'_\mu\}$ is related to $\{e_\mu\}$ by the Lorentz transformation A.

Example. Take D to be the vector representation. Then the carrier space is just $\mathbb{R}^n$ so we can denote the resulting vector ψ^μ. Then $D(A) = A$ and so under a change of basis we have $\psi'^{\mu} = A^{\mu}{}_{\nu}\psi^{\nu}$, which is just the usual transformation law for the components of a vector.

Remark. Given a field transforming in some representation D, we can take a partial derivative of its components in a coordinate basis, and then convert the result to our orthonormal basis, i.e., $\partial_\mu \psi \equiv e^{\alpha}_{\mu}\partial_\alpha \psi$ where α refers to the coordinate basis. However, since the matrix A can depend on position, the partial derivative of our field will no longer transform homogeneously under a Lorentz transformation: the result will involve derivatives of A. For tensor fields, we know how to resolve this problem: introduce the covariant derivative. So now we need to extend the definition of covariant derivative to a general representation D:

Definition. The covariant derivative of a field ψ transforming in a representation of the Lorentz group with generators $T^{\mu\nu}$ is, in an orthonormal basis,

$$\nabla_\mu \psi = \partial_\mu \psi + \frac{1}{2}\left(\omega_{\nu\rho}\right)_\mu T^{\nu\rho}\psi \tag{9.41}$$

Remark. One can show that this does indeed transform correctly, i.e., in a representation of the Lorentz group, under Lorentz transformations. The connection 1-forms are sometimes referred to as the *spin connection* because of their role in defining the covariant derivative for spinor fields.

Definition. The *Dirac equation* for a spin 1/2 field of mass m is

$$\Gamma^\mu \nabla_\mu \psi - m\psi = 0. \tag{9.42}$$

9.4 Curvature 2-forms

Definition. Consider a spacetime with an orthonormal basis. The *curvature 2-forms* are

$$\Theta^{\mu}{}_{\nu} = \frac{1}{2}R^{\mu}{}_{\nu\rho\sigma}e^{\rho}\wedge e^{\sigma} \tag{9.43}$$

Remark. The antisymmetry of the Riemann tensor implies $\Theta_{\mu\nu} = -\Theta_{\nu\mu}$.

Lemma. The curvature 2-forms are given in terms of the connection 1-forms by

$$\Theta^{\mu}{}_{\nu} = d\omega^{\mu}{}_{\nu} + \omega^{\mu}{}_{\rho}\wedge\omega^{\rho}{}_{\nu} \tag{9.44}$$

Proof. Optional exercise. Direct calculation of the RHS, using the relation (9.17) and equation (9.19). You'll need to work out the generalization of (6.6) to a non-coordinate basis.

Remark. This provides a convenient way of calculating the Riemann tensor in an orthonormal basis. One calculates the connection 1-forms using (9.19) and then the curvature 2-forms using (9.43). The components of $\Theta^\mu{}_\nu$ are $R^\mu{}_{\nu\rho\sigma}$ so one can read off the Riemann tensor components. The only derivatives one needs to calculate are exterior derivatives, which are usually fairly easy. In simple situations, this is much faster than calculating the Riemann tensor in a coordinate basis using (6.6).

Example. We determined the connection 1-forms in the Schwarzschild spacetime previously. From $\omega^0{}_1 = f'e^0$ we have

$$d\omega^0{}_1 = f'de^0 + f''dr \wedge e^0 = f'^2 e^1 \wedge e^0 + ff''e^1 \wedge e^0 \tag{9.45}$$

we also have

$$\omega^0{}_\rho \wedge \omega^\rho{}_1 = \omega^0{}_1 \wedge \omega^1{}_1 = 0 \tag{9.46}$$

where the first equality arises because $\omega^0{}_\rho$ is non-zero only for $\rho = 1$ and the second equality is because $\omega^1{}_1 = \omega_{11} = 0$ (by antisymmetry). Combining these results we have

$$\Theta_{01} = -\Theta^0{}_1 = \left(ff'' + f'^2\right) e^0 \wedge e^1 \tag{9.47}$$

and hence the only non-vanishing components of the form $R_{01\rho\sigma}$ are

$$R_{0101} = -R_{0110} = \left(ff'' + f'^2\right) = \frac{1}{2}\left(f^2\right)'' = -\frac{2M}{r^3} \tag{9.48}$$

Exercise (examples sheet 4). Determine the remaining curvature 2-forms Θ_{02}, Θ_{03}, Θ_{12}, Θ_{13}, Θ_{23} (all others are related to these by (9.44)). Hence determine the Riemann tensor components. Check that the Ricci tensor vanishes.

9.5 Volume form

Definition. A manifold of dimension n is *orientable* if it admits an *orientation*: a smooth, nowhere vanishing n-form $\epsilon_{a_1 \dots a_n}$. Two orientations ϵ and ϵ' are *equivalent* if $\epsilon' = f\epsilon$ where f is an everywhere positive function.

Remark. Any n-form X is related to ϵ by $X = f\epsilon$ for some function f. X will define an orientation provided f does not vanish anywhere. Hence an orientable manifold admits precisely two inequivalent orientations, corresponding to the cases $f > 0$ and $f < 0$.

Definition. A coordinate chart x^μ on an orientable manifold is *right-handed* with respect to ϵ iff $\epsilon = f(x)dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$ with $f(x) > 0$. It is *left-handed* if $f(x) < 0$.

Definition. On an oriented manifold with a metric, the *volume form* is defined by

$$\epsilon_{12\dots n} = \sqrt{|g|} \tag{9.49}$$

in any right-handed coordinate chart, where g denotes the determinant of the metric in this chart.

Exercise. 1. Show that this definition is chart-independent. 2. Show that (in a RH coordinate chart)

$$\epsilon^{12\dots n} = \pm \frac{1}{\sqrt{|g|}} \tag{9.50}$$

where the upper (lower) sign holds for Riemannian (Lorentzian) signature.

Lemma.

$$\epsilon^{a_1\dots a_p c_{p+1}\dots c_n} \epsilon_{b_1\dots b_p c_{p+1}\dots c_n} = \pm p!(n-p)! \delta^{a_1}_{[b_1} \dots \delta^{a_p}_{b_p]} \tag{9.51}$$

where the upper (lower) sign holds for Riemannian (Lorentzian) signature.
Proof. Exercise.

Definition. On an oriented manifold with metric, the *Hodge dual* of a p-form X is the $(n-p)$-form $\star X$ defined by

$$(\star X)_{a_1\dots a_{n-p}} = \frac{1}{p!} \epsilon_{a_1\dots a_{n-p} b_1 \dots b_p} X^{b_1\dots b_p} \tag{9.52}$$

Lemma. For a p-form X,

$$\star(\star X) = \pm(-1)^{p(n-p)} X \tag{9.53}$$

$$(\star d \star X)_{a_1\dots a_{p-1}} = \pm(-1)^{p(n-p)} \nabla^b X_{a_1\dots a_{p-1} b} \tag{9.54}$$

where the upper (lower) sign holds for Riemannian (Lorentzian) signature.
Proof. Exercise (use (9.51)).

Examples.

1. In 3d Euclidean space, the usual operations of vector calculus can be written using differential forms as

$$\nabla f = df \qquad \text{div}\, X = \star d \star X \qquad \text{curl}\, X = \star dX \tag{9.55}$$

   where f is a function and X denotes the 1-form X_a dual to a vector field X^a. The final equation shows that the exterior derivative can be thought of as a generalization of the curl operator.

2. Maxwell's equations are $\nabla^a F_{ab} = -4\pi j_b$ and $\nabla_{[a}F_{bc]} = 0$ where j^a is the current density vector. These can be written as

$$d \star F = -4\pi \star j, \qquad dF = 0 \tag{9.56}$$

The first of these implies $d \star j = 0$, which is equivalent to $\nabla_a j^a = 0$, i.e., j^a is a conserved current. The second of these implies (by the Poincaré lemma) that *locally* there exists a 1-form A such that $F = dA$.

9.6 Integration on manifolds

Definition. Let M be an oriented manifold of dimension n. Let $\psi : \mathcal{O} \to \mathcal{U}$ be a RH coordinate chart with coordinates x^μ and let X be a n-form. The *integral of* X *over* $\mathcal{O}$ is

$$\int_{\mathcal{O}} X \equiv \int_{\mathcal{U}} dx^1 \dots dx^n X_{12\dots n} \tag{9.57}$$

Exercise. Show that this is chart independent, i.e., if one replaces ϕ with another RH coordinate chart $\phi' : \mathcal{O} \to \mathcal{U}'$ then one gets the same result.

Remark. How do we extend our definition to all of M? The idea is just to chop M up into regions such that we can use the above definition on each region, then sum the resulting terms.

Definition. Let the RH charts $\phi_\alpha : \mathcal{O}_\alpha \to \mathcal{U}_\alpha$ be an atlas for M. Introduce a "partition of unity", i.e., a set of functions $h_\alpha : M \to [0, 1]$ such that $h_\alpha(p) = 0$ if $p \notin \mathcal{O}_\alpha$, and $\sum_\alpha h_\alpha(p) = 1$ for all p. We then define

$$\int_M X \equiv \sum_\alpha \int_{\mathcal{O}_\alpha} h_\alpha X \tag{9.58}$$

Remark.

1. It can be shown that this definition is independent of the choice of atlas and partition of unity.

2. A diffeomorphism $\phi : M \to M$ is *orientation preserving* if $\phi^*(\epsilon)$ is equivalent to ϵ for any orientation ϵ. It is not hard to show that the integral is invariant under orientation preserving diffeomorphisms:

$$\int_M \phi^*(X) = \int_M X \tag{9.59}$$

Definition. Let M be an oriented manifold with a metric g. Let ϵ be the volume form. The *volume* of M is $\int_M \epsilon$. If f is a function on M then

$$\int_M f \equiv \int_M f\epsilon \tag{9.60}$$

Remark. We shall sometimes use the notation

$$\int_M f = \int_M d^n x \sqrt{|g|} f \tag{9.61}$$

This is an abuse of notation because the RHS refers to coordinates x^μ but M might not be covered by a single chart. It has the advantage of making it clear that the integral depends on the metric tensor.

9.7 Submanifolds and Stokes' theorem

Definition. Let S and M be oriented manifolds of dimension m and n respectively, $m < n$. A smooth map $\phi : S \to M$ is an *embedding* if it is one-to-one ($\phi(p) \neq \phi(q)$ for $p \neq q$, i.e. $\phi[S]$ does not intersect itself) and for any $p \in S$ there exists a neighbourhood $\mathcal{O}$ such that $\phi^{-1} : \phi[\mathcal{O}] \to S$ is smooth. If ϕ is an embedding then $\phi[S]$ is an *embedded submanifold of* M. A *hypersurface* is an embedded submanifold of dimension $n - 1$.

Remark. The technical details here are included for completeness, we won't need to refer to them again. Henceforth, we will simply say "submanifold" instead of "embedded submanifold".

Definition. If $\phi[S]$ is a m-dimensional submanifold of M and X is a m-form on M then the integral of X over $\phi[S]$ is

$$\int_{\phi[S]} X \equiv \int_S \phi^*(X) \tag{9.62}$$

Remark. If $X = dY$ then the fact that d and ϕ^* commute gives

$$\int_{\phi[S]} dY = \int_S d\left(\phi^*(Y)\right) \tag{9.63}$$

Definition. A *manifold with boundary* M is defined in the same way as a manifold except that charts are maps $\phi_\alpha : \mathcal{O}_\alpha \to \mathcal{U}_\alpha$ where now $\mathcal{U}_\alpha$ is an open subset of $\frac{1}{2}\mathbb{R}^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^1 \leq 0\}$. The boundary of M, denoted ∂M, is the set of points for which $x^1 = 0$. This is a manifold of dimension $n-1$ with coordinate charts $(x^2, \ldots x^n)$. If M is oriented then the orientation of ∂M is fixed by saying that $(x^2, \ldots x^n)$ is a RH chart on ∂M when $(x^1, \ldots, x^n)$ is a RH chart on M.

Stokes' theorem. Let N be an oriented n-dimensional manifold with boundary and X a $(n-1)$-form. Then

$$\int_N dX = \int_{\partial N} X \tag{9.64}$$

Remarks. We define the RHS by regarding ∂N as a hypersurface in N (the map ϕ is just the inclusion map) and using (9.62). We often use this result when N is some region of a larger manifold M. Then ∂N is a hypersurface in M.

Example. Let Σ be a hypersurface in a spacetime M and consider a solution of Maxwell's equations (9.56)). Assume that Σ has a boundary. Then

$$\frac{1}{4\pi}\int_{\partial\Sigma} \star F = \frac{1}{4\pi}\int_{\Sigma} d\star F = -\int_{\Sigma} \star j \equiv Q[\Sigma] \tag{9.65}$$

The final equality defines the total charge on Σ. Hence we have a formula relating the charge on Σ to the flux through the boundary of Σ. This is Gauss' law.

Definition. $X \in T_p(M)$ is *tangent to* $\phi[S]$ *at* p if X is the tangent vector at p of a curve that lies in $\phi[S]$. $n \in T_p^*(M)$ is *normal* to a submanifold $\phi[S]$ if $n(X) = 0$ for any vector X tangent to $\phi[S]$ at p.

Remark. The vector space of tangent vectors to $\phi[S]$ at p has dimension m. The vector space of normals to $\phi[S]$ at p has dimension $n-m$. Any two normals to a hypersurface are proportional to each other.

Definition. A hypersurface in a Lorentzian manifold is *timelike*, *spacelike* or *null* if any normal is everywhere spacelike, timelike or null respectively.

Remark. Let M is a manifold with boundary and consider a curve in ∂M with parameter t and tangent vector X. Then $x^1(t) = 0$ so

$$dx^1(X) = X(x^1) = \frac{dx^1}{dt} = 0. \tag{9.66}$$

Hence $dx^1(X)$ vanishes for any X tangent to ∂M so dx^1 is normal to ∂M. Any other normal to ∂M will be proportional to dx^1. If ∂M is timelike or spacelike then we can construct a *unit* normal by dividing by the norm of dx^1:

$$n_a = \frac{(dx^1)_a}{\sqrt{\pm g^{bc}(dx^1)_b(dx^1)_c}} \qquad \Rightarrow \qquad g^{ab}n_a n_b = \pm 1 \tag{9.67}$$

One can show that this is chart independent. Here we choose the + sign if dx^1 is spacelike and the $-$ sign if dx^1 is timelike (+ if the metric is Riemannian). Note that n^a "points out of" M if ∂M is timelike (or the metric is Riemannian) but into M if ∂M is spacelike. This is to get the correct sign in the divergence theorem:

Divergence theorem. If ∂M is timelike or spacelike then

$$\int_M d^n x \sqrt{|g|}\, \nabla_a X^a = \int_{\partial M} d^{n-1} x \sqrt{|h|}\, n_a X^a \tag{9.68}$$

where X^a is a vector field on M, ∇ is the Levi-Civita connection, and h denotes the determinant of the metric on ∂M induced by pulling back the metric on M. $n_a X^a$ is a scalar in M so it can be pulled back to ∂M, this is the integrand on the RHS.

Proof. Follow through the definitions, using (9.54), (9.53) and Stokes' theorem. The volume form of ∂M is $\hat{\epsilon}$ where $\hat{\epsilon}_{2\ldots n} = \sqrt{|h|}$ in one of the coordinate charts occuring in the definition of a manifold with boundary. In such a chart, the components $h_{\mu\nu}$ are the same as the components $g_{\mu\nu}$ with $2 \leq \mu, \nu \leq n$. Using this, $g^{ab}(dx^1)_a (dx^1)_b = g^{11} = h/g$.

Chapter 10

Lagrangian formulation

10.1 Scalar field action

You are familiar with the idea that the equation of motion of a point particle can be obtained by extremizing an action. You may also know that the same is true for fields in Minkowski spacetime. The same is true in GR. To see how this works, consider first a scalar field, i.e., a function $\Phi : M \to \mathbb{R}$ and define the *action* as the functional

$$S[\Phi] = \int_M d^4x \, \sqrt{-g} L \tag{10.1}$$

where L is the *Lagrangian*:

$$L = -\frac{1}{2} g^{ab} \nabla_a \Phi \nabla_b \Phi - V(\Phi) \tag{10.2}$$

and $V(\Phi)$ is called the scalar potential. Now consider a variation $\Phi \to \Phi + \delta\Phi$ for some function $\delta\Phi$ that vanishes on ∂M (in an asymptotically flat spacetime, ∂M will be "at infinity"). The change in the action is (working to linear order in $\delta\Phi$)

$$\begin{aligned}
\delta S &= S[\Phi + \delta\Phi] - S[\Phi] \\
&= \int_M d^4x \, \sqrt{-g} \left(-g^{ab} \nabla_a \Phi \nabla_b \delta\Phi - V'(\Phi)\delta\Phi \right) \\
&= \int_M d^4x \, \sqrt{-g} \left[-\nabla_a \left(\delta\Phi \nabla^a \Phi \right) + \delta\Phi \nabla^a \nabla_a \Phi - V'(\Phi)\delta\Phi \right] \\
&= \int_{\partial M} d^3x \, \sqrt{|h|} \, \delta\Phi \, n_a \nabla^a \Phi + \int_M d^4x \, \sqrt{-g} \left(\nabla^a \nabla_a \Phi - V'(\Phi) \right) \delta\Phi \\
&= \int_M d^4x \, \sqrt{-g} \left(\nabla^a \nabla_a \Phi - V'(\Phi) \right) \delta\Phi
\end{aligned} \tag{10.3}$$

Note that we have used the divergence theorem to "integrate by parts". A formal way of writing the final expression is

$$\delta S = \int_M d^4x \frac{\delta S}{\delta \Phi} \delta \Phi \tag{10.4}$$

where

$$\frac{\delta S}{\delta \Phi} \equiv \sqrt{-g} \left(\nabla^a \nabla_a \Phi - V'(\Phi) \right) \tag{10.5}$$

The factor of $\sqrt{-g}$ here means that this quantity is not a scalar (it is an example of a *scalar density*). However $(1/\sqrt{-g})\delta S/\delta \Phi$ is a scalar. We've written things in this strange way in order to be consistent with how we treat the gravitational field.

Demanding that δS vanishes for arbitrary $\delta \Phi$ gives us the equation of motion $\delta S/\delta \Phi = 0$, i.e.,

$$\nabla^a \nabla_a \Phi - V'(\Phi) = 0. \tag{10.6}$$

The particular choice $V(\Phi) = \frac{1}{2} m^2 \Phi^2$ gives the Klein-Gordon equation.

10.2 The Einstein-Hilbert action

For the gravitational field, we seek an action of the form

$$S[g] = \int_M d^4x \sqrt{-g} L \tag{10.7}$$

where L is a scalar constructed from the metric. An obvious choice for the Lagrangian is $L \propto R$. This gives the *Einstein-Hilbert action*

$$S_{EH}[g] = \frac{1}{16\pi} \int_M d^4x \sqrt{-g} R = \frac{1}{16\pi} \int_M R\epsilon \tag{10.8}$$

where the prefactor is included for later convenience and ϵ is the volume form. The idea is that we regard our manifold M as fixed (e.g. $\mathbb{R}^4$) and g_{ab} is determined by extremizing $S[g]$. In other words, we consider two metrics g_{ab} and $g_{ab} + \delta g_{ab}$ and demand that $S[g + \delta g] - S[g]$ should vanish to linear order in δg_{ab}. Note that δg_{ab} is the difference of two metrics and hence is a tensor field.

We need to work out what happens to ϵ and R when we vary $g_{\mu\nu}$. Recall the formula for the determinant "expanding along the μth row":

$$g = \sum_\nu g_{\mu\nu} \Delta^{\mu\nu} \tag{10.9}$$

where we are suspending the summation convention, μ is any fixed value, and $\Delta^{\mu\nu}$ is the cofactor matrix, whose $\mu\nu$ element is $(-1)^{\mu+\nu}$ times the determinant of the matrix obtained by deleting row μ and column ν from the metric. Note that $\Delta^{\mu\nu}$ is independent of the $\mu\nu$ element of the metric. Hence

$$\frac{\partial g}{\partial g_{\mu\nu}} = \Delta^{\mu\nu} = g g^{\mu\nu} \tag{10.10}$$

where on the RHS we recall the formula for the inverse matrix $g^{\mu\nu}$ in terms of the cofactor matrix. We can use this formula to determine how g varies under a small change $\delta g_{\mu\nu}$ in $g_{\mu\nu}$ (reinstating the summation convention):

$$\delta g = \frac{\partial g}{\partial g_{\mu\nu}} \delta g_{\mu\nu} = g g^{\mu\nu} \delta g_{\mu\nu} = g g^{ab} \delta g_{ab} \tag{10.11}$$

(we can use abstract indices in the final equality since $g^{ab}\delta g_{ab}$ is a scalar) and hence

$$\delta \sqrt{-g} = \frac{1}{2} \sqrt{-g}\, g^{ab} \delta g_{ab} \tag{10.12}$$

From the definition of the volume form we have

$$\delta \epsilon = \frac{1}{2} \epsilon g^{ab} \delta g_{ab} \tag{10.13}$$

Next we need to evaluate δR. To this end, consider first the change in the Christoffel symbols. $\delta \Gamma^\mu_{\nu\rho}$ is the difference between the components of two connections (i.e. the Levi-Civita connections associated to $g_{ab} + \delta g_{ab}$ and g_{ab}). Since the difference of two connections is a tensor, it follows that $\delta \Gamma^\mu_{\nu\rho}$ are components of a tensor $\delta \Gamma^a_{bc}$. These components are easy to evaluate if we introduce normal coordinates at p for the unperturbed connection: at p we have $g_{\mu\nu,\rho} = 0$ and $\Gamma^\mu_{\nu\rho} = 0$. For the perturbed connection we therefore have, at p, (to linear order)

$$\begin{aligned} \delta \Gamma^\mu_{\nu\rho} &= \frac{1}{2} g^{\mu\sigma} \left(\delta g_{\sigma\nu,\rho} + \delta g_{\sigma\rho,\nu} - \delta g_{\nu\rho,\sigma} \right) \\ &= \frac{1}{2} g^{\mu\sigma} \left(\delta g_{\sigma\nu;\rho} + \delta g_{\sigma\rho;\nu} - \delta g_{\nu\rho;\sigma} \right) \end{aligned} \tag{10.14}$$

In the second equality, the semi-colon denotes a covariant derivative with respect to the Levi-Civita connection associated to g_{ab}. The two expressions are equal because $\Gamma(p) = 0$. The LHS and RHS are tensors so this is a basis independent result hence we can use abstract indices:

$$\delta \Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\delta g_{db;c} + \delta g_{dc;b} - \delta g_{bc;d} \right) \tag{10.15}$$

p is arbitrary so this result holds everywhere.

Now consider the variation of the Riemann tensor. Again it is convenient to use normal coordinates at p, so at p we have (using $\delta(\Gamma\Gamma) \sim \Gamma\delta\Gamma = 0$ at p)

$$\begin{aligned} \delta R^{\mu}{}_{\nu\rho\sigma} &= \partial_\rho \delta\Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \delta\Gamma^{\mu}_{\nu\rho} \\ &= \nabla_\rho \delta\Gamma^{\mu}_{\nu\sigma} - \nabla_\sigma \delta\Gamma^{\mu}_{\nu\rho} \end{aligned} \tag{10.16}$$

where ∇ is the Levi-Civita connection of g_{ab}. Once again we can immediately replace the basis indices by abstract indices:

$$\delta R^{a}{}_{bcd} = \nabla_c \delta\Gamma^{a}_{bd} - \nabla_d \delta\Gamma^{a}_{bc} \tag{10.17}$$

and p is arbitrary so the result holds everywhere. Contracting gives the variation of the Ricci tensor:

$$\delta R_{ab} = \nabla_c \delta\Gamma^{c}_{ab} - \nabla_b \delta\Gamma^{c}_{ac} \tag{10.18}$$

Finally we have

$$\delta R = \delta(g^{ab} R_{ab}) = g^{ab}\delta R_{ab} + \delta g^{ab} R_{ab} \tag{10.19}$$

where δg^{ab} is the variation in g^{ab} (*not* the result of raising indices on δg_{ab}). Using $\delta(g_{\mu\rho} g^{\rho\nu}) = \delta(\delta^{\nu}_{\mu}) = 0$ it is easy to show (exercise)

$$\delta g^{ab} = -g^{ac} g^{bd} \delta g_{cd} \tag{10.20}$$

Putting everything together, we have

$$\begin{aligned} \delta R &= -g^{ac} g^{bd} R_{ab} \delta g_{cd} + g^{ab} \left(\nabla_c \delta\Gamma^{c}_{ab} - \nabla_b \delta\Gamma^{c}_{ac} \right) \\ &= -R^{ab}\delta g_{ab} + \nabla_c (g^{ab} \delta\Gamma^{c}_{ab}) - \nabla_b (g^{ab}\delta\Gamma^{c}_{ac}) \\ &= -R^{ab}\delta g_{ab} + \nabla_a X^a \end{aligned} \tag{10.21}$$

where

$$X^a = g^{bc} \delta\Gamma^{a}_{bc} - g^{ab}\delta\Gamma^{c}_{bc} \tag{10.22}$$

Hence the variation of the Einstein-Hilbert action is

$$\begin{aligned} \delta S_{EH} &= \frac{1}{16\pi} \int_M \delta(\epsilon\, R) \\ &= \frac{1}{16\pi} \int_M \epsilon \left(\frac{1}{2} R g^{ab} \delta g_{ab} - R^{ab}\delta g_{ab} + \nabla_a X^a \right) \\ &= \frac{1}{16\pi} \int_M d^4x\, \sqrt{-g} \left(\frac{1}{2} R g^{ab} \delta g_{ab} - R^{ab}\delta g_{ab} + \nabla_a X^a \right) \end{aligned} \tag{10.23}$$

The final term can be converted to a surface term on ∂M using the divergence theorem. If we assume that δg_{ab} has support in a compact region that doesn't intersect ∂M then this term will vanish (because vanishing of δg_{ab} and its derivative on ∂M implies that X^a will vanish on ∂M). Hence we have

$$\delta S_{EH} = \frac{1}{16\pi}\int_M d^4x\sqrt{-g}\left(-G^{ab}\right)\delta g_{ab} = \int_M \frac{\delta S_{EH}}{\delta g_{ab}}\delta g_{ab} \tag{10.24}$$

where G_{ab} is the Einstein tensor and

$$\frac{\delta S_{EH}}{\delta g_{ab}} = -\frac{1}{16\pi}\sqrt{-g}G^{ab} \tag{10.25}$$

Hence extremization of S_{EH} reproduces the vacuum Einstein equation.

Exercise. Show that the vacuum Einstein equation with cosmological constant is obtained by extremizing

$$S_{EH\Lambda} = \frac{1}{16\pi}\int_M d^4x\sqrt{-g}\,(R - 2\Lambda) \tag{10.26}$$

Remark. The *Palatini procedure* is a different way of deriving the Einstein equation from the Einstein-Hilbert action. Instead of using the Levi-Civita connection, we allow for an arbitrary torsion-free connection. The EH action is then a functional of both the metric and the connection, which are to be varied independently. Varying the metric gives the Einstein equation (but written with an arbitrary connection). Varying the connection implies that the connection should be the Levi-Civita connection. When matter is included, this works only if the matter action is independent of the connection (as is the case for a scalar field or Maxwell field) or if the Levi-Civita connection is used in the matter action.

10.3 Energy momentum tensor

Next we consider the action for matter. We assume that this is given in terms of the integral of a scalar Lagrangian:

$$S_{\text{matter}} = \int d^4x\sqrt{-g}L_{\text{matter}} \tag{10.27}$$

here L_{matter} is a function of the matter fields (assumed to be tensor fields), their derivatives, the metric and its derivatives. An example is given by the scalar field Lagrangian discussed above. We define the energy momentum tensor by

$$T^{ab} = \frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g_{ab}} \tag{10.28}$$

in other words, under a variation in g_{ab} we have (after integrating by parts using the divergence theorem to eliminate derivatives of δg_{ab} if present)

$$\delta S_{\text{matter}} = \frac{1}{2}\int_M d^4x\sqrt{-g}T^{ab}\delta g_{ab} \tag{10.29}$$

This definition clearly makes T^{ab} symmetric.

Example. Consider the scalar field action we discussed previously.

$$S = \int_M \epsilon L \tag{10.30}$$

with L given by (10.2). Using the results for $\delta\epsilon$ and δg^{ab} derived above we have, under a variation of g_{ab}:

$$\delta S = \int_M d^4x\sqrt{-g}\left[\frac{1}{2}\nabla^a\Phi\nabla^b\Phi + \frac{1}{2}\left(-\frac{1}{2}g^{cd}\nabla_c\Phi\nabla_d\Phi - V(\Phi)\right)g^{ab}\right]\delta g_{ab} \tag{10.31}$$

Hence

$$T^{ab} = \nabla^a\Phi\nabla^b\Phi + \left(-\frac{1}{2}g^{cd}\nabla_c\Phi\nabla_d\Phi - V(\Phi)\right)g^{ab} \tag{10.32}$$

If we define the total action to be $S_{EH} + S_{\text{matter}}$ then under a variation of g_{ab} we have

$$\frac{\delta}{\delta g_{ab}}\left(S_{EH} + S_{\text{matter}}\right) = \sqrt{-g}\left(-\frac{1}{16\pi}G^{ab} + \frac{1}{2}T^{ab}\right) \tag{10.33}$$

and hence demanding that $S_{EH} + S_{\text{matter}}$ be extremized under variation of the metric gives the Einstein equation

$$G_{ab} = 8\pi T_{ab} \tag{10.34}$$

How do we know that our definition of T_{ab} gives a conserved tensor? It follows from the fact that S_{matter} is diffeomorphism invariant. In more detail, diffeomorphisms are a gauge symmetry so the total action $S = S_{EH} + S_{\text{matter}}$ should be diffeomorphism invariant in the sense that $S[g,\Phi] = S[\phi_*(g),\phi_*(\Phi)]$ where Φ denotes the matter fields and ϕ is a diffeomorphism. The Einstein-Hilbert action alone is diffeomorphism invariant. Hence S_{matter} also must be diffeomorphism invariant. The easiest way of ensuring this is to take it to be the integral of a scalar Lagrangian as we assumed above.

Now consider the effect of an infinitesimal diffeomorphism. As we saw when discussing linearized theory (eq (8.13)), an infinitesimal diffeomorphism shifts g_{ab} by

$$\delta g_{ab} = \mathcal{L}_\xi g_{ab} = \nabla_a\xi_b + \nabla_b\xi_a \tag{10.35}$$

Matter fields also transform according to the Lie derivative (eq (8.12)), e.g. for a scalar field:

$$\delta\Phi = \mathcal{L}_\xi \Phi = \xi^a \nabla_a \Phi \tag{10.36}$$

Let's consider this scalar field case in detail. Assume that the matter Lagrangian is an arbitrary scalar constructed from Φ, the metric, and arbitrarily many of their derivatives (e.g. there could be a term of the form $\nabla_a \nabla_b \Phi \nabla^a \nabla^b \Phi$ or $R\Phi^2$). Under an infinitesimal diffeomorphism, (after integration by parts to remove derivatives from $\delta\Phi$ and δg_{ab})

$$\begin{aligned} \delta S_{\text{matter}} &= \int_M d^4x \left(\frac{\delta S_{\text{matter}}}{\delta \Phi} \delta\Phi + \frac{\delta S_{\text{matter}}}{\delta g_{ab}} \delta g_{ab} \right) \\ &= \int_M d^4x \left(\frac{\delta S_{\text{matter}}}{\delta \Phi} \xi^b \nabla_b \Phi + \frac{1}{2}\sqrt{-g} T^{ab} \delta g_{ab} \right) \end{aligned} \tag{10.37}$$

The second term can be written

$$\begin{aligned} \int_M d^4x \sqrt{-g} T^{ab} \nabla_a \xi_b &= \int_M d^4x \sqrt{-g} \left[\nabla_a \left(T^{ab} \xi_b \right) - \left(\nabla_a T^{ab} \right) \xi_b \right] \\ &= -\int_M d^4x \sqrt{-g} \left(\nabla_a T^{ab} \right) \xi_b \end{aligned} \tag{10.38}$$

where we assume that ξ_b vanishes on ∂M so the total derivative can be discarded. Now diffeomorphism invariance implies that δS_{matter} must vanish for arbitrary ξ_b. Hence we must have

$$\frac{\delta S_{\text{matter}}}{\delta \Phi} \nabla^b \Phi - \sqrt{-g} \nabla_a T^{ab} = 0. \tag{10.39}$$

Hence we see that if the scalar field equation of motion ($\delta S_{\text{matter}}/\delta\Phi = 0$) is satisfied then

$$\nabla_a T^{ab} = 0. \tag{10.40}$$

This is a special case of a very general result. Diffeomorphism invariance plus the equations of motion for the matter fields implies energy-momentum tensor conservation. It applies for a matter Lagrangian constructed from tensor fields of any type (the matter fields), the metric, and arbitrarily many derivatives of the matter fields and metric.

An identical argument applied to the Einstein-Hilbert action leads to the contracted Bianchi identity (exercise):

$$\nabla_a G^{ab} = 0. \tag{10.41}$$

Hence the contracted Bianchi identity is a consequence of diffeomorphism invariance of the Einstein-Hilbert action.