Exercises in Relativity and Cosmology II summer term 2016

Problem 60

Show that a rotational surface in $\mathbf{E}^3$ can be parametrized as

$$\vec{X}(l,\phi) = (f(l)cos\phi, f(l)sin\phi, g(l)) \text{ with } f'^2 + g'^2 = 1.$$

Compute with this parametrization the induced metric and the Gaussian curvature K of the surface. Result: $K = -f''/f$.

Problem 61

Assume $K = -1$ and show that the simplest solution for the surface is determined by the differential equation

$$\frac{dZ}{dR} = -\frac{\sqrt{1-R^2}}{R}$$

with $R^2 = X^2 + Y^2$. Sketch the surface qualitatively and conclude from this that a surface of constant negative curvature cannot be embedded in $\mathbf{E}^3$ without singularities.

Problem 62

a) Show that the metric of 60. can be transformed to

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$$

with (x, y) in the upper half plane. Is this manifold complete?

b) The same geometry can also be obtained from embedding the surface in (2+1)-dimensional Minkowski space with coordinates X^0, X^1, X^2. Show that the parametrization

$$y^i = \frac{X^i}{1+X^0}, \ i = 1, 2$$

yields the metric

$$ds^2 = \frac{4\sum(dy^i)^2}{(1-\sum(y^k)^2)^2}.$$

What is the geometrical meaning of this parametrization?

Problem 63

Show for an isotropic homogeneous universe:

The projection (along the distinguished wordlines) of a null geodesic on a homogeneity hypersurface Σ is

a) a geodesic with respect to the 3-geometry of Σ,

b) a Killing trajectory.

Problem 64

Sketch the radius $r(t)$ of the past lightcone with apex at time t_0. Calculate the time t_{max} of its maximal extension for a scale factor $R(t) \propto t^\alpha$, $0 < \alpha < 1$, and the corresponding value r_{max}.

Problem 65

The *particle horizon distance* $d_H(t)$ is defined as the maximal distance of objects visible (in principle) at cosmic time t. Calculate $d_H(t)$ for a scale factor $R(t) \propto t^\alpha,\ 0 < \alpha < 1$. What is the velocity $v_H(t)$ of a comoving object at this distance?

Problem 66

An observer's *future event horizon* is defined as the hypersurface from beyond which she will never receive any signal. What is the condition on the scale factor $R(t)$ required for this horizon to exist?

Problem 67

Represent the distance function $d_0(z)$ as an integral using $z(t)$ as integration variable instead of t. Use the Friedmann equation to represent $\dot{R}/R$ as a function of z with given cosmological parameters Ω_m and Ω_Λ. Represent in a similar way the emission time $t(z)$ and the age of the universe T assuming that Ω_m and Ω_Λ provide a complete description of its contents.

Problem 68

The *de Sitter universe* is the maximally symmetric spacetime realized by the hyperboloid $\{X_0^2 - \sum_1^4 X_i^2 = -R^2\}$ in 5-dimensional Minkowski space. Show that the metric of this spacetime can take the following forms:

a) $ds^2 = dt^2 - R^2 cosh^2(t/R)(\frac{dr^2}{1-r^2} + r^2 d\Omega^2)$,

b) $ds^2 = dt^2 - R^2 sinh^2(t/R)(\frac{dr^2}{1+r^2} + r^2 d\Omega^2)$,

c) $ds^2 = dt^2 - R^2 e^{2t/R}(dr^2 + r^2 d\Omega^2)$.

Use the result of 62.b) to write down the manifestly conformally flat form of the metric. What is the source term in the Einstein equations for the de Sitter solution?