Exercises in Relativity and Cosmology II summer term 2016

Problem 35

Calculate the deficit vector Δ of the infinitesimal parallelogram defined by two vectors on an affine manifold.

Problem 36

Show that the coefficients of a torsion-free connection can be transformed to zero at an arbitrary point.

Problem 37

Show that the coefficients of a torsion-free connection can be transformed to zero along an arbitrary line.
Optional: In which sense is this statement true if the torsion does not vanish?

Problem 38

Calculate $\nabla_{[i}\nabla_{j]}f$ and $\nabla_{[i}\nabla_{j]}v^a$.

Problem 39

One of the earliest attempts to geometrize electrodynamics endowed Minkowski space with the connection defined by the Cartesian coefficients $L^i_{jk} = \delta^i{}_j A_k$.
a) Compute the curvature tensor of this connection.
b) Compute $\nabla_i \eta_{jk}$.
c) Why is this connection physically not acceptable? *Hint*: Consider two clocks that become separated and are then brought together again.
d) Compute the connection coefficients in curvilinear coordinates and for a general Lorentz metric.

Problem 40

Referred to an anholonomic basis $\{e_A = e_A^{\,i}\partial_i\}$ with dual $\{e^B\}$, the connection coefficients of 39. become

$$L^A{}_{Bi} = e^A{}_j e_B{}^k L^j{}_{ki} + e^A{}_l e_B{}^l{}_{,i}$$

(why?). For which choice of $\{e_A\}$ is this transformation equivalent to a gauge transformation $A_i \mapsto A_i + \Lambda_{,i}$?.

Problem 41

The 1-form of a Riemann-Cartan connection referred to an orthonormal tetrad $\{e_A\}$ is called spin connection. Show that every spin connection fulfils $L_{ABi} = -L_{BAi}$.

Problem 42

a) Modify the Levi-Civita parallel transport of a vector v along an *arbitrary* worldline in such a way that the velocity vector is reproduced, but the other properties of parallel transport (linearity and conservation of the scalar product) are maintained. *Hint*: The transport equation has to contain the absolute derivative of v, the 4-velocity u and the 4-acceleration a.

b) Show that this modified transport implies the Thomas precession in Minkowski space. (No calculation required!)

Problem 43

The Theorema egregium. A smooth surface in $\mathbb{E}^3$ can in the vicinity of an arbitrary point be approximately described by the equation $z = \frac{x^2}{2R_1} + \frac{y^2}{2R_2}$ (why?). Show that R_1 and R_2 are the principal curvature radii at the point being considered. Compute the induced metric $g_{ij}(x, y)$ in the vicinity of and the curvature scalar at this point.

Problem 44

Prove the formula for the sectional curvature of a geodesic surface

a) for $n = 2$;

b) for $n > 2$. (*Hint*: The intrinsic Riemann tensor at the origin of the geodesic surface is identical to the restriction of the n-dimensional Riemann tensor.)

Problem 45

Prove the identity $R_{ijkl} = R_{klij}$ as a consequence of the identities 1.-3. listed for the Riemann tensor in the lecture.

Problem 46

Prove the algebraic symmetries of the Weyl tensor and the vanishing of its trace.

Problem 47

Verify the formula for the components of a metric volume form in local coordinates.

Problem 48

In 4 dimensions the covariant Riemann tensor can be dualised with respect to both the first and second index pair. Show that the doubly dualised Riemann tensor fulfils $(*R*)_{ki}{}^{jk} = G_i{}^j$.

Problem 49

Show

a) $\delta g = g g^{ik} \delta g_{ik}$ and

b) $\frac{1}{\sqrt{|g|}} \partial_i \sqrt{|g|} = \Gamma^l{}_{il}$.

c) Conclude from b) that the covariant divergence of an antisymmetric tensor field T is given by

$$\nabla_i T^{i i_1 \cdots i_p} = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} T^{i i_1 \cdots i_p} \right).$$