Exercises in Relativity and Cosmology II summer term 2016

Problem 26
The Maxwell equations in the Lorenz gauge follow from the Lagrangian \( \mathcal{L} = \frac{1}{8\pi} A_i \epsilon^{i,k} - j^i A_i \). Show that the canonical energy-momentum tensor of the electromagnetic field derived from this Lagrangian coincides with the Maxwell tensor in the case of a plane wave in the radiation gauge.

Problem 27
Construct a Lagrangian for the linearized gravitational field \( \psi_{ik} \) in the harmonic gauge analogous to the Lagrangian of Problem 25 (with interaction term \( -\psi_{ik} T^{ik} \)) and compute from it the energy-momentum tensor of a plane wave in the TT gauge. Compare the result with that derived in the lecture.

Problem 28
Estimate the energy-current density of a gravitational wave with amplitude \( h \sim 10^{-20} \) and frequency \( \sim 100\text{Hz} \) and compare it with that of moonlight.

Problem 29
What is the gravitational radiation power of a satellite of mass \( m \) in circular orbit around a central mass \( M \) at a distance \( r \)? What is the frequency of the radiation? Provide the numbers for the example of the earth orbiting the sun.

Problem 30
Derive an upper limit for the radiation power of any bound system (use \( V_0 \leq c^2 \)).

Problem 31
Let \( M_1 \) be the real line \( \mathbb{R} \) with the standard differentiability structure and \( M_2 \) the differentiable manifold defined by \( \mathbb{R} \) with the atlas \( \{ x \mapsto x^3 \} \). Show:

a) The differentiability structures of \( M_1 \) and \( M_2 \) are not compatible.

b) The identity on \( \mathbb{R} \) is not a diffeomorphism from \( M_1 \) to \( M_2 \).

c) Give an example of a diffeomorphism from \( M_1 \) to \( M_2 \).

Problem 32
Show that the abstract definition of a tangent vector \( v \) implies that \( v(c) = 0 \) for a constant function \( c \).

Problem 33
Show that the tangent vectors \( \partial_i|_p \) form a basis for \( T_p \). Use and prove the following lemma: An arbitrary function \( \phi \in C^\infty(\mathbb{R}^n) \) can be written as \( \phi(x) = \phi(0) + g_i(x)x^i \) with \( g_i(0) = \partial_i\phi|_0 \).

Problem 34
Show abstractly that the Lie bracket of two vector fields is a vector field and compute the components of \( [u, v] \).