Exercises in Relativity and Cosmology II summer term 2015

Problem 49

Show $\mathcal{L}_u v = [u, v]$ and $[\mathcal{L}_u, \mathcal{L}_v] = \mathcal{L}_{[u,v]}$.

Problem 50

Show: If there exists a nontrivial Killing vector field ξ, then there exist coordinates such that g_{ik} does not depend on one of them.

Problem 51

The Gauss theorem for an antisymmetric tensor field J^{ik} defined on a hypersurface Σ with boundary reads

$$\int_\Sigma \nabla_i J^{ik} d\sigma_k = -\frac{1}{2} \int_{\partial\Sigma} J^{ik} d\sigma_{ik}$$

$$(d\sigma_{ik} = \frac{1}{2} \epsilon_{iklm} dx^l \wedge dx^m).$$

Use this theorem to show that a spacetime with Killing vector ξ possesses the conserved quantity

$$Q_\xi := -\frac{1}{\kappa} \int_{S_\infty} \nabla^i \xi^k d\sigma_{ik},$$

where S_∞ is the 'boundary at infinity' of the spacelike hypersurface Σ. Use the Einstein field equations to find a relation between this conserved quantity and $\int_\Sigma T^{ik} \xi_i d\sigma_k$. What is Q_ξ in the case of the timelike Killing vector of the Schwarzschild metric?

Problem 52

Generalize the special-relativistic Maxwell action

$$S[A] = -\frac{1}{16\pi} \int d^4x F_{ij} F^{ij} + \int d^4x j^i A_i$$

to a curved spacetime and derive from this generalization the covariant Maxwell equations.

Problem 53

The GPS satellites are in a circular orbit 20200 km above the surface of the earth. A GPS receiver at rest on the surface of the earth compares the received satellite clock signals with its own clock. What is the ratio of corresponding time intervals? Estimate from this the error in position determination that would arise in the course of a day from neglecting relativistic corrections.

Problem 54

Show that every 3-dimensional spherically symmetric metric is conformally flat.

Problem 55

Prove Kepler's third law for circular orbits in the Schwarzschild metric: $\omega^2 r^3 = \mathcal{M}$, where $\omega = d\phi/dt$ and r is the Schwarzschild radial coordinate.

Problem 56

Compute the delay (in earth time) of the radar echo of a planet (at distance r_1 from the sun) a) in opposition or lower conjunction, b) in upper conjunction with the sun.

Problem 57

Show that for a static spherically symmetric star with $\epsilon = const$ $p(0) \to \infty$ for $R \to \frac{9}{4}\mathcal{M}$.

Problem 58

Estimate the mass limit for a star consisting of relativistic bosons.

Problem 59

A mass m is lowered by a rope towards the Schwarzschild horizon and held stationary there. What is the force that has to be exerted on the other end of the rope (far away from the horizon)? Hint: $k = dE/dl$.

Problem 60

Compute the proper time for the radial fall in the extended Schwarzschild spacetime from $r = R$ to $r = 0$.

Problem 61

Show for an isotropic homogeneous universe:
The projection (along the distinguished wordlines) of a null geodesic on a homogeneous hypersurface Σ is
a) a geodesic with respect to the 3-geometry of Σ,
b) a Killing trajectory.

Problem 62

Show that in an isotropic homogeneous universe with scale factor $R(t) \propto t^\alpha,\ 0 < \alpha < 1$, the present distance from an object with redshift z is

$$d_0 = \frac{\alpha}{1-\alpha} H_0^{-1}[1 - (1+z)^{1-1/\alpha}].$$