Exercises in Relativity and Cosmology II summer term 2015

Problem 1
In Newton’s theory a static and spatially constant gravitational field can be transformed to zero. For the relativistic generalization of this statement, construct a family of observers in 2-dimensional Minkowski spacetime moving in such a manner that everyone experiences a constant acceleration and the distances between them (as measured by themselves) remain constant. Show that the worldlines of the observers are concentric equilateral hyperbolas (setting $c = 1$). What is the main difference between this and the corresponding Newtonian family?

Problem 2
Introduce coordinates $\tau, \xi$ adapted to the observers of Problem 1 so that the hyperbolas and the lines of simultaneity for these observers are coordinate lines. Show that the coordinates can be chosen such that the Minkowski line element reads $ds^2 = \xi^2 d\tau^2 - d\xi^2$.

Problem 3
According to Newton’s theory a thin massive plate of infinite extension generates a constant gravitational field on both sides of the plate. Try to draw a spacetime diagram for the relativistic analog of this configuration using a) coordinates of the type introduced in Problem 2, b) Cartesian coordinates.

Problem 4
Compute the line element of Minkowski space in a reference frame $\Sigma'$ that rotates with constant angular velocity $\Omega$ about the z-axis of an inertial frame $\Sigma$. Introduce cylindrical coordinates in the inertial frame and then use $\phi' = \phi - \Omega t$ with the other coordinates unchanged. The result is (with the primes removed)

$ds^2 = \left(1 - \Omega^2(x^2 + y^2)\right) dt^2 + 2\Omega(ydx - xdy) dt - dx^2 - dy^2 - dz^2$.

Problem 5
Write down the geodesic equations for $x, y$ and $z$ in the rotating system and show that in the nonrelativistic limit they reduce to Newton’s equation of motion for a free particle in a rotating system.

Problem 6
Show that the distance travelled (as measured by an inertial observer) is a possible affine parameter for a massless particle in Minkowski space. Try to generalize this statement to an arbitrary spacetime.

Problem 7
Show that the equation

$\ddot{x}^i + \Gamma^i_{kl} \dot{x}^k \dot{x}^l = \kappa(x, \dot{x}, \cdots) \dot{x}^i$

with an ‘arbitrary’ function $\kappa$ and the derivatives referring to a parameter $\lambda$ describes a geodesic. Do this by constructing an affine parameter $\tau = \tau(\lambda)$ with the help of $\kappa$. 
Problem 8
Verify the transformation property of the Christoffel symbols:
\[ \Gamma^{\alpha}_{\quad jk} = \frac{\partial x^{\alpha}}{\partial x^{i}} \frac{\partial x^{m}}{\partial x^{j}} \frac{\partial x^{n}}{\partial x^{k}} \Gamma^{i}_{\quad mn} + \frac{\partial^{2} x^{i}}{\partial x^{j} \partial x^{k}} \frac{\partial x^{i}}{\partial x^{l}}. \]

Conclude from this the validity of the formula for \( \nabla_{k} v^{i} \).

Problem 9
Verify the formula \( \nabla_{k} v_{i} = v_{i,k} - \Gamma^{l}_{\quad ik} v_{l} \).

Problem 10
a) Show that the following equation holds in the origin of a Riemannian normal coordinate system:
\[ \Gamma^{i}_{\quad jk,l} + \Gamma^{i}_{\quad lj,k} + \Gamma^{i}_{\quad kl,j} = 0. \]

Hint: Consider \( d^{3}x^{i}/d\tau^{3} \) along geodesics.

b) Prove the cyclic symmetry of the Riemann tensor.

Problem 11
Conclude from 10.a) that in the origin of a RNCS
\[ g_{ij,kl} = \frac{1}{3} (R_{iklj} + R_{ilkj}). \]

Problem 12
Show that \( R_{abcd} \) is determined by \( S_{abcd} := R_{d(ab)c} \). Hint: Antisymmetrize \( S_{abcd} \) in \( b \) and \( c \).