Exercises in Relativity and Cosmology II summer term 2015

Problem 1
In Newton's theory a static and spatially constant gravitational field can be transformed to zero. For the relativistic generalization of this statement, construct a family of observers in 2-dimensional Minkowski spacetime moving in such a manner that everyone experiences a constant acceleration and the distances between them (as measured by themselves) remain constant. Show that the worldlines of the observers are concentric equilateral hyperbolas (setting $c = 1$). What is the main difference between this and the corresponding Newtonian family?

Problem 2
Introduce coordinates τ, ξ adapted to the observers of Problem 1 so that the hyperbolas and the lines of simultaneity for these observers are coordinate lines. Show that the coordinates can be chosen such that the Minkowski line element reads $ds^2 = \xi^2 d\tau^2 - d\xi^2$.

Problem 3
According to Newton's theory a thin massive plate of infinite extension generates a constant gravitational field on both sides of the plate. Try to draw a spacetime diagram for the relativistic analog of this configuration using a) coordinates of the type introduced in Problem 2, b) Cartesian coordinates.

Problem 4
Compute the line element of Minkowski space in a reference frame Σ' that rotates with constant angular velocity Ω about the z-axis of an inertial frame Σ. Introduce cylindrical coordinates in the inertial frame and then use $\phi' = \phi - \Omega t$ with the other coordinates unchanged. The result is (with the primes removed)
$ds^2 = (1 - \Omega^2(x^2 + y^2))dt^2 + 2\Omega(ydx - xdy)dt - dx^2 - dy^2 - dz^2$.

Problem 5
Write down the geodesic equations for x, y and z in the rotating system and show that in the nonrelativistic limit they reduce to Newton's equation of motion for a free particle in a rotating system.

Problem 6
Show that the distance travelled (as measured by an inertial observer) is a possible affine parameter for a massless particle in Minkowski space. Try to generalize this statement to an arbitrary spacetime.

Problem 7
Show that the equation

$$\ddot{x}^i + \Gamma^i{}_{kl}\dot{x}^k\dot{x}^l = \kappa(x, \dot{x}, \cdots)\dot{x}^i$$

with an 'arbitrary' function κ and the derivatives referring to a parameter λ describes a geodesic. Do this by constructing an affine parameter $\tau = \tau(\lambda)$ with the help of κ.

Problem 8

Verify the transformation property of the Christoffel symbols:

$$\Gamma'^{i}{}_{jk} = \frac{\partial x'^i}{\partial x^l} \frac{\partial x^m}{\partial x'^j} \frac{\partial x^n}{\partial x'^k} \Gamma^l{}_{mn} + \frac{\partial x'^i}{\partial x^l} \frac{\partial^2 x^l}{\partial x'^j \partial x'^k}.$$

Conclude from this the validity of the formula for $\nabla_k v^i$.

Problem 9

Verify the formula $\nabla_k v_i = v_{i,k} - \Gamma^l{}_{ik} v_l$.

Problem 10

a) Show that the following equation holds in the origin of a Riemannian normal coordinate system:

$$\Gamma^i{}_{jk,l} + \Gamma^i{}_{lj,k} + \Gamma^i{}_{kl,j} = 0.$$

Hint: Consider $d^3x^i/d\tau^3$ along geodesics.

b) Prove the cyclic symmetry of the Riemann tensor.

Problem 11

Conclude from 10.a) that in the origin of a RNCS

$$g_{ij,kl} = \frac{1}{3}(R_{iklj} + R_{ilkj}).$$

Problem 12

Show that R_{abcd} is determined by $S_{abcd} := R_{d(ab)c}$. Hint: Antisymmetrize S_{abcd} in b and c.