Exercises in Relativity and Cosmology II summer term 2014

Problem 46
Show $\mathcal{L}_u v = [u, v]$ and $[\mathcal{L}_u, \mathcal{L}_v] = \mathcal{L}_{[u,v]}$.

Problem 47
Show: If there exists a nontrivial Killing vector field ξ, then there exist coordinates such that g_{ik} does not depend on one of them.

Problem 48
The Gauss theorem for an antisymmetric tensor field J^{ik} defined on a hypersurface Σ with boundary reads

$$\int_\Sigma \nabla_i J^{ik} d\sigma_k = -\frac{1}{2}\int_{\partial\Sigma} J^{ik} d\sigma_{ik}$$

$$(d\sigma_{ik} = \pm\frac{1}{2}\epsilon_{iklm} dx^l \wedge dx^m).$$

Use this theorem to show that a spacetime with Killing vector ξ possesses the conserved quantity

$$\int_{S_\infty} \nabla^i \xi^k d\sigma_{ik},$$

where S_∞ is the 'boundary at infinity' of the spacelike hypersurface Σ. Use the Einstein field equations to find a relation between this conserved quantity and $\int_\Sigma T^{ik}\xi_i d\sigma_k$.

Problem 49
Generalize the special-relativistic Maxwell action

$$S[A] = -\frac{1}{16\pi}\int d^4x F_{ij}F^{ij} + \int d^4x j^i A_i$$

to a curved spacetime and derive from this generalization the covariant Maxwell equations.

Problem 50
The GPS satellites are in a circular orbit 20200 km above the surface of the earth. A GPS receiver at rest on the surface of the earth compares the received satellite clock signals with its own clock. What is the ratio of corresponding time intervals? Estimate from this the error in position determination that would arise in the course of a day from neglecting relativistic corrections.

Problem 51
Show that every 3-dimensional spherically symmetric metric is conformally flat.

Problem 52
Prove Kepler's third law for circular orbits in the Schwarzschild metric: $\omega^2 r^3 = \mathcal{M}$, where $\omega = d\phi/dt$ and r is the Schwarzschild radial coordinate.

Problem 53

Show that the capture cross section for massless particles in the Schwarzschild geometry is $\sigma_c = 27\pi\mathcal{M}^2$.

Problem 54

Compute the delay (in earth time) of the radar echo of a planet (at distance r_1 from the sun) a) in opposition or lower conjunction, b) in upper conjunction with the sun.

Problem 55

Show that for a static spherically symmetric star with $\epsilon = const$ $p(0) \to \infty$ for $R \to \frac{9}{4}\mathcal{M}$.

Problem 56

Estimate the mass limit for a star consisting of relativistic bosons.

Problem 57

Compute the proper time for the radial fall in the extended Schwarzschild spacetime from $r = R$ to $r = 0$.