Exercises in Relativity and Cosmology II summer term 2014

Problem 25
Estimate
a) the order of magnitude of $\Omega_m$ on the surface of the earth,
b) the order of magnitude of $\Omega_{stat}$ for a low earth orbit.

Problem 26
Show that Fermi-Walker transport in Minkowski space implies the Thomas precession. 
*Hint:* The proof is short and does not require the calculation of the Thomas frequency.

Problem 27
Conclude from 26. that spin precession in Minkowski space is described by the transport equation
\[
\frac{d\vec{s}}{dt} = \gamma^2 \vec{v} \left( \frac{d\vec{v}}{dt} \cdot \vec{s} \right)
\]
and derive from the rotational part of the r.h.s. of this equation the nonrelativistic limit of the Thomas frequency
\[
\vec{\Omega}_{Th} \approx -\frac{1}{2} \vec{v} \times \frac{d\vec{v}}{dt}
\]

*Optional:* How would one derive the exact Thomas frequency from the transport equation?

Problem 28
Let $M_1$ be the real line $\mathbb{R}$ with the standard differentiability structure and $M_2$ the differentiable manifold defined by $\mathbb{R}$ with the atlas $\{x \mapsto x^3\}$. Show:
a) The differentiability structures of $M_1$ and $M_2$ are not compatible.
b) The identity on $\mathbb{R}$ is not a diffeomorphism from $M_1$ to $M_2$.
c) Give an example of a diffeomorphism from $M_1$ to $M_2$.

Problem 29
Show that the abstract definition of a tangent vector $v$ implies that $v(c) = 0$ for a constant function $c$.

Problem 30
Show that the tangent vectors $\partial_i|_P$ form a basis for $T_P$. Use and prove the following lemma: An arbitrary function $\phi \in C^\infty(\mathbb{R}^n)$ can be written as $\phi(x) = \phi(0) + g_i(x)x^i$ with $g_i(0) = \partial_i \phi|_0$.

Problem 31
Show abstractly that the Lie bracket of two vector fields is a vector field and compute the components of $[u,v]$. 