Problem 1
In Newton’s theory a static and spatially constant gravitational field can be transformed to zero. For the relativistic generalization of this statement, construct a family of observers in 2-dimensional Minkowski spacetime moving in such a manner that everyone experiences a constant acceleration and the distances between them (as measured by themselves) remain constant. Show that the worldlines of the observers are concentric equilateral hyperbolas (setting \( c = 1 \)). What is the main difference between this and the corresponding Newtonian family?

Problem 2
Introduce coordinates \( \tau, \xi \) adapted to the observers of Problem 1 so that the hyperbolas and the lines of simultaneity for these observers are coordinate lines. Show that the coordinates can be chosen such that the Minkowski line element reads \( ds^2 = \xi^2 d\tau^2 - d\xi^2 \).

Problem 3
According to Newton’s theory a thin massive plate of infinite extension generates a constant gravitational field on both sides of the plate. Try to draw a spacetime diagram for the relativistic analog of this configuration using a) coordinates of the type introduced in Problem 2, b) Cartesian coordinates.

Problem 4
Compute the line element of Minkowski space in a reference frame \( \Sigma' \) that rotates with constant angular velocity \( \Omega \) about the z-axis of an inertial frame \( \Sigma \). Introduce cylindrical coordinates in the inertial frame and then use \( \phi' = \phi - \Omega t \) with the other coordinates unchanged. The result is (with the primes removed)
\[
ds^2 = (1 - \Omega^2(x^2 + y^2))dt^2 + 2\Omega(ydx - xdy)dt - dx^2 - dy^2 - dz^2.
\]

Problem 5
Write down the geodesic equations for \( x, y \) and \( z \) in the rotating system and show that in the nonrelativistic limit they reduce to Newton’s equation of motion for a free particle in a rotating system.

Problem 6
Show that the distance travelled (as measured by an inertial observer) is a possible affine parameter for a massless particle in Minkowski space. Try to generalize this statement to an arbitrary spacetime.

Problem 7
Show that the equation
\[
\ddot{x}^i + \Gamma^i_{kli}\dot{x}^k\dot{x}^l = \kappa(x, \dot{x}, \cdots)\dot{x}^i
\]
with an 'arbitrary' function \( \kappa \) and the derivatives referring to a parameter \( \lambda \) describes a geodesic. Do this by constructing an affine parameter \( \tau = \tau(\lambda) \) with the help of \( \kappa \).
Problem 8
Verify the transformation property of the Christoffel symbols:

$$\Gamma^i_{jk} = \frac{\partial x^l}{\partial x^i} \frac{\partial x^m}{\partial x^j} \frac{\partial x^n}{\partial x^k} \Gamma^l_{mn} + \frac{\partial x^l}{\partial x^i} \frac{\partial^2 x^l}{\partial x^j \partial x^k}. $$

Conclude from this the validity of the formula for $\nabla_k v^i$.

Problem 9
Verify the formula $\nabla_k v_i = v_{i,k} - \Gamma^l_{ik} v_l$.

Problem 10
a) Show that the following equation holds in the origin of a Riemannian normal coordinate system:

$$\Gamma^i_{jk,l} + \Gamma^i_{lj,k} + \Gamma^i_{kl,j} = 0. $$

Hint: Consider $d^3x^i/d\tau^3$ along geodesics.

b) Prove the cyclic symmetry of the Riemann tensor.

Problem 11
Conclude from 10.a) that in the origin of a RNCS

$$g_{ij,kl} = \frac{1}{3}(R_{iklj} + R_{ilkj}).$$

Problem 12
Show that $R_{abcd}$ is determined by $S_{abcd} := R_{d(ab)c}$. Hint: Antisymmetrize $S_{abcd}$ in $b$ and $c$. 