1 Show that

$$[\mathcal{L}_{\mathcal{X}}, \mathcal{L}_{\mathcal{Y}}] = \mathcal{L}_{[\mathcal{X},\mathcal{Y}]} \ .$$

2 Let Σ be given by the equation $x^1 = f(x^2, \ldots, x^n)$. The *metric induced on* Σ *by a metric* g is defined as the metric obtained by replacing, in g, all occurrences of x^1 by f and of dx^1 by df. So, for example, $d\Omega^2$ is the metric induced on the unit sphere by the flat metric $dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\Omega^2$ on $\mathbb{R}^3$, since on S^2 one substitutes $r = 1$ and $dr = 0$.

Let h be the metric on constant time slices in a spherically symmetric static star with constant density ρ. Show that h is the metric on a three sphere of a radius which you should determine. *[Hint: write the sphere* $S^3 \subset \mathbb{R}^4$ *as* $w = \sqrt{R^2 - x^2 - y^2 - z^2} = \sqrt{R^2 - r^2}$*, and calculate the metric induced on* S^3 *by the Euclidean metric* $dx^2 + dy^2 + dz^2 + dw^2 = dr^2 + r^2 d\Omega^2 + dw^2$*.]*

3 Calculate the total gravitational potential energy of a spherically symmetric Newtonian star with mass density ρ which is constant within a ball of radius R. *[Hint: To start, find the Newton potential* ϕ *of the configuration, solution of*

$$\Delta\phi = 4\pi G\rho \ ,$$

satisfying $\lim_{r\to\infty} \phi = 0$*. Here you can use the fact that the solution must be spherically symmetric (why?), so that*

$$\Delta\phi = \frac{1}{r^2}\partial_r(r^2\partial_r\phi) \ .$$

You can then use the fact that a shell at radius r of thickness dr will contribute

$$dU = 4\pi\rho r^2 \phi dr$$

to the total potential energy U of the star.]