1. Show that
\[ [\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]} . \]

2. Let \( \Sigma \) be given by the equation \( x^1 = f(x^2, \ldots, x^n) \). The metric induced on \( \Sigma \) by a metric \( g \) is defined as the metric obtained by replacing, in \( g \), all occurrences of \( x^1 \) by \( f \) and of \( dx^1 \) by \( df \). So, for example, \( d\Omega^2 \) is the metric induced on the unit sphere by the flat metric \( dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\Omega^2 \) on \( \mathbb{R}^3 \), since on \( S^2 \) one substitutes \( r = 1 \) and \( dr = 0 \).

Let \( h \) be the metric on constant time slices in a spherically symmetric static star with constant density \( \rho \). Show that \( h \) is the metric on a three sphere of a radius which you should determine. [Hint: write the sphere \( S^3 \subset \mathbb{R}^4 \) as \( w = \sqrt{R^2 - x^2 - y^2 - z^2} = \sqrt{R^2 - r^2} \), and calculate the metric induced on \( S^3 \) by the Euclidean metric \( dx^2 + dy^2 + dz^2 + dw^2 = dr^2 + r^2 d\Omega^2 + dw^2 \).]

3. Calculate the total gravitational potential energy of a spherically symmetric Newtonian star with mass density \( \rho \) which is constant within a ball of radius \( R \). [Hint: To start, find the Newton potential \( \phi \) of the configuration, solution of
\[ \Delta \phi = 4\pi G\rho , \]
satisfying \( \lim_{r \to \infty} \phi = 0 \). Here you can use the fact that the solution must be spherically symmetric (why?), so that
\[ \Delta \phi = \frac{1}{r^2} \partial_r (r^2 \partial_r \phi) . \]
You can then use the fact that a shell at radius \( r \) of thickness \( dr \) will contribute
\[ dU = 4\pi \rho r^2 \phi dr \]
to the total potential energy \( U \) of the star.]