1 QI. The Lane-Emden equation

Recall from lectures that, for a Newtonian static fluid, p and ρ satisfy the equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi\rho.$$

Assume that $\rho = \rho_c\theta(r)^n$ and $p = p_c\theta(r)^{n+1}$, where n, ρ_c, p_c are constants, and θ is a function to be determined. Check that

$$p = K\rho^{\frac{n+1}{n}},$$

for a constant K that you will determine. Show that the function θ satisfies the differential equation

$$\frac{K(n+1)}{4\pi}\rho_c^{\frac{1}{n}-1}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n.$$

Let $r_n > 0$ be defined by

$$r_n^2 = \frac{K(n+1)}{4\pi}\rho_c^{\frac{1}{n}-1},$$

and define a new variable ξ by $r = r_n\xi$. Deduce that

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n.$$

Solve this equation with boundary conditions $\theta(0) = 1$, $\theta'(0) = 0$ in the case $n = 0$. In the case $n = 1$, let $\theta(\xi) = \psi(\xi)/\xi$, and solve for $\psi(\xi)$ with the same boundary conditions for θ as in the case $n = 0$. Finally, verify that the function

$$\theta(\xi) = \frac{1}{\left(1 + \frac{1}{3}\xi^2\right)^{1/2}}$$

is a solution in the case $n = 5$.

2 Given a one-form, α, we define its exterior derivative, $d\alpha$, as

$$d\alpha_{ab} = \partial_a\alpha_b - \partial_b\alpha_a.$$

Show that $d\alpha$ is a tensor.

Show that, acting on arbitrary vector fields X, Y, we have

$$d\alpha(X, Y) = \mathcal{L}_X(\alpha(Y)) - \mathcal{L}_Y(\alpha(X)) - \alpha([X, Y]).$$

(Note that this allows us to define the exterior derivative of a one-form in terms of the Lie derivative.) Also, show that for arbitrary vector fields X, Y, we have

$$(\mathcal{L}_X\alpha)(Y) = d\alpha(X, Y) + Y(\alpha(X)).$$

Recall that if f is a function, then $df = \partial_i f dx^i$. Show that $d(df) = 0$, and

$$\mathcal{L}_X(df) = d(\mathcal{L}_X f).$$