1 A vector field X^μ is called *Killing* if

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0 \; .$$

i. Check that ∂_t and ∂_φ are Killing vectors both in the Minkowski metric and in the Schwarzschild metric.

ii. Show that if γ is a geodesic and X is Killing, then $g(\dot\gamma, X)$ is constant along γ.

iii. Show that if $\nabla_\mu T^\mu{}_\nu = 0$ and X is Killing then the vector field $J^\mu := T^\mu{}_\nu X^\nu$ has vanishing divergence, $\nabla_\mu J^\mu = 0$.

2 Let φ satisfy the wave equation in a general space-time with Lorentzian metric g, $\Box_g \varphi := \nabla^\mu \nabla_\mu \varphi = 0$. Set

$$T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} \nabla^\alpha \varphi \partial_\alpha \varphi g_{\mu\nu} \; .$$

Show that $\nabla_\mu T^\mu{}_\nu = 0$.

Let $X = \partial_0$ be a Killing vector. Given a hypersurface $\mathscr{S} = \{x^0 = \text{const.}\}$, with unit future-directed timelike normal n^μ, set

$$E := \int_{\mathscr{S}} T_{\mu\nu} X^\nu n^\mu \sqrt{\det g_{ij}} d^3 x \; .$$

Here $g_{ij} dx^i dx^j$ is the space-part of the metric, where one sets to zero all terms involving $g_{0\mu}$. E is called the total energy of the field contained in $\mathscr{S}$. Give an explicit expression for E for the surface $\{t = 0\}$ in the Minkowski space-time, and in the Schwarzschild space-time.

3 <u>Lie derivative</u>

Given a vector field X, the *Lie derivative* $\mathcal{L}_X$ is an operation on tensors, defined as follows:

i. For a function f, one sets $\mathcal{L}_X f := X(f)$.

ii. For a vector field Y, one sets $\mathcal{L}_X Y := [X, Y]$, the Lie bracket.

iii. For a one form α, $\mathcal{L}_X \alpha$ is defined by imposing the Leibniz rule written backwards:

$$(\mathcal{L}_X \alpha)(Y) := \mathcal{L}_X(\alpha(Y)) - \alpha(\mathcal{L}_X Y) \; .$$

Q3.1: a) Why is this the same as the Leibniz rule? *[Hint: write this equation using indices.]* b) Check that $\mathcal{L}_X \alpha$ is a tensor. *[Hint: check that the right-hand-side is linear under multiplication of Y by a function.]*

Q3.2: Show that

$$(\mathcal{L}_X \alpha)_a = X^b \partial_b \alpha_a + \alpha_b \partial_a X^b \; .$$

For tensor products the Lie derivative is defined again by imposing linearity under addition together with the Leibniz rule:

$$\mathcal{L}_X(\alpha \otimes \beta) = (\mathcal{L}_X \alpha) \otimes \beta + \alpha \otimes \mathcal{L}_X \beta \ .$$

Since a general tensor A is sum of tensor products,

$$A = A^{a_1 \ldots a_p}{}_{b_1 \ldots b_q} \partial_{a_1} \otimes \ldots \partial_{a_p} \otimes dx^{b_1} \otimes \ldots \otimes dx^{b_q} \ ,$$

requiring linearity with respect to addition of tensors gives thus a definition of Lie derivative for any tensor.

Q3.3: Show that

$$\mathcal{L}_X T^a{}_b = X^c \partial_c T^a{}_b - T^c{}_b \partial_c X^a + T^a{}_c \partial_b X^c \ ,$$

Q3.4: Can you see the general formula for the Lie derivative $\mathcal{L}_X A^{a_1 \ldots a_p}{}_{b_1 \ldots b_q}$?