1. A vector field $X^\mu$ is called Killing if
   \[ \nabla_\mu X_\nu + \nabla_\nu X_\mu = 0 \].
   i. Check that $\partial_t$ and $\partial_\phi$ are Killing vectors both in the Minkowski metric and in the Schwarzschild metric.
   ii. Show that if $\gamma$ is a geodesic and $X$ is Killing, then $g(\dot{\gamma}, X)$ is constant along $\gamma$.
   iii. Show that if $\nabla_\mu T^\mu_\nu = 0$ and $X$ is Killing then the vector field $J^\mu := T^\mu_\nu X^\nu$ has vanishing divergence, $\nabla_\mu J^\mu = 0$.

2. Let $\varphi$ satisfy the wave equation in a general space-time with Lorentzian metric $g$, $\Box_g \varphi := \nabla^\mu \nabla_\mu \varphi = 0$. Set
   \[ T^\mu_\nu = \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} \nabla^\alpha \varphi \partial_\alpha \varphi g^\mu_\nu. \]
   Show that $\nabla_\mu T^\mu_\nu = 0$.
   Let $X = \partial_0$ be a Killing vector. Given a hypersurface $\mathcal{S} = \{x^0 = \text{const.}\}$, with unit future-directed timelike normal $n^\mu$, set
   \[ E := \int_{\mathcal{S}} T^\mu_\nu X^\nu n^\mu \sqrt{\det g_{ij}} \, dx. \]
   Here $g_{ij} dx^i dx^j$ is the space-part of the metric, where one sets to zero all terms involving $g_{0\mu}$. $E$ is called the total energy of the field contained in $\mathcal{S}$. Give an explicit expression for $E$ for the surface $\{t = 0\}$ in the Minkowski space-time, and in the Schwarzschild space-time.

3. Lie derivative
   Given a vector field $X$, the Lie derivative $\mathcal{L}_X$ is an operation on tensors, defined as follows:
   i. For a function $f$, one sets $\mathcal{L}_X f := X(f)$.
   ii. For a vector field $Y$, one sets $\mathcal{L}_X Y := [X, Y]$, the Lie bracket.
   iii. For a one form $\alpha$, $\mathcal{L}_X \alpha$ is defined by imposing the Leibniz rule written backwards:
   \[ (\mathcal{L}_X \alpha)(Y) := \mathcal{L}_X (\alpha(Y)) - \alpha(\mathcal{L}_X Y). \]
   Q3.1: a) Why is this the same as the Leibniz rule? [Hint: write this equation using indices.] b) Check that $\mathcal{L}_X \alpha$ is a tensor. [Hint: check that the right-hand-side is linear under multiplication of $Y$ by a function.]
   Q3.2: Show that
   \[ (\mathcal{L}_X \alpha)_a = X^b \partial_b \alpha_a + \alpha_b \partial_a X^b. \]
For tensor products the Lie derivative is defined again by imposing linearity under addition together with the Leibniz rule:

\[ \mathcal{L}_X (\alpha \otimes \beta) = (\mathcal{L}_X \alpha) \otimes \beta + \alpha \otimes \mathcal{L}_X \beta. \]

Since a general tensor \( A \) is sum of tensor products,

\[ A = A^{a_1 \ldots a_p}_{b_1 \ldots b_q} \partial_{a_1} \otimes \ldots \otimes \partial_{a_p} \otimes dx^{b_1} \otimes \ldots \otimes dx^{b_q}, \]

requiring linearity with respect to addition of tensors gives thus a definition of Lie derivative for any tensor.

Q3.3: Show that

\[ \mathcal{L}_X T^a_{\ b} = X^c \partial_c T^a_{\ b} - T^c_{\ b} \partial_c X^a + T^a_{\ c} \partial_b X^c, \]

Q3.4: Can you see the general formula for the Lie derivative \( \mathcal{L}_X A^{a_1 \ldots a_p}_{b_1 \ldots b_q} \)?