1 Alice circles planet X freely for a long time on a circular orbit of radius R, while her twin Bob remains motionless on the surface of the planet, at radius r_0. For $r \geq r_0$ the geometry of the gravitational field of the planet X is described by the Schwarzschild metric with mass $0 < m < r_0/2$. Derive a necessary and sufficient condition on R which guarantees that, on meeting Bob again, Alice will have the same age as Bob. You should assume that the time of travel back and forth from radius R to radius r_0 can be neglected compared to the time that Alice spent on the circular orbit.

2 Recall that in one of the previous problem sheets we have derived the identity

$$\nabla^\mu \nabla_\mu f = \frac{1}{\sqrt{|\det g_{\alpha\beta}|}} \partial_\mu \left(\sqrt{|\det g_{\alpha\beta}|} g^{\mu\nu} \partial_\nu f \right) .$$

i. Show that, for weak gravitational fields $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu}$ appropriately small, the *wave coordinates* condition

$$\nabla^\mu \nabla_\mu x^\alpha = 0$$

approximately reads

$$\partial_\mu h^\mu{}_\nu = \frac{1}{2} \partial_\nu (h^\alpha{}_\alpha) ,$$

where $h^\alpha{}_\beta = \eta^{\alpha\gamma} h_{\gamma\beta}$.

3 Let $h_{\alpha\beta} = \Re(A_{\alpha\beta} \exp(ik_\mu x^\mu))$, where $\Re$ denotes the real part, be a linearized gravitational wave in TT gauge (i.e., $A^\alpha{}_\alpha = 0 = A_{\alpha\beta} k^\beta$). Show that, in the linear approximation,

$$R_{\alpha\beta\gamma\delta} k^\delta = 0 .$$

4 Let φ satisfy the wave equation in a general space-time with Lorentzian metric g, $\Box_g \varphi := \nabla^\mu \nabla_\mu \varphi = 0$. Set

$$T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} \nabla^\alpha \varphi \partial_\alpha \varphi g_{\mu\nu} .$$

Show that $T_{00} \geq \sqrt{\sum_i (T_{0i})^2}$. Show that this is equivalent to the statement that for any future-pointing timelike vector u^μ, the vector $T^\mu{}_\nu u^\mu$ is timelike past-pointing.

Show that T satisfies the *dominant energy condition*: $T_{\mu\nu} X^\mu Y^\nu \geq 0$ for all timelike future directed X and Y. Is this condition satisfied for the energy-momentum tensor of dust?