1 Isotropic coordinates for the Schwarzschild metric. We wish to find coordinates $(t, \tilde{r}, \theta, \phi)$, where $\tilde{r} = \tilde{r}(r)$, for the Schwarzschild metric in terms of which the spatial metric is conformally flat. In particular, we wish to find functions $\tilde{r}(r)$, $A(\tilde{r})$ and $B(\tilde{r})$ such that

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \\ &= -A(\tilde{r})^2 dt^2 + B(\tilde{r})^2 (d\tilde{r}^2 + \tilde{r}^2 d\Omega^2). \end{aligned} \tag{1}$$

Q1. Deduce that we require

$$\begin{aligned} A(\tilde{r})^2 &= 1 - \frac{2m}{r}, \\ B(\tilde{r})^2 \left(\frac{d\tilde{r}}{dr}\right)^2 &= \frac{r}{r - 2m}, \\ B(\tilde{r})^2 \tilde{r}^2 &= r^2. \end{aligned} \tag{2}$$

Q2. Assuming that $d\tilde{r}/dr > 0$ and $\tilde{r} > 0$, show that $\tilde{r}$ satisfies the relation

$$\frac{1}{\tilde{r}} \frac{d\tilde{r}}{dr} = \frac{1}{\sqrt{r(r - 2m)}}. \tag{3}$$

Integrating with respect to r from $r = 2m$ and letting $\tilde{r}(2m) = m/2$, show that

$$\tilde{r}(r) = \frac{m}{2} \left(\frac{r - m}{m} + \sqrt{\left(\frac{r - m}{m}\right)^2 - 1} \right) \tag{4}$$

and

$$r = \frac{1}{\tilde{r}} \left(\tilde{r} + \frac{m}{2}\right)^2. \tag{5}$$

Q3. Show that

$$A(\tilde{r})^2 = \left(\tilde{r} + \frac{m}{2}\right)^2, \quad B(\tilde{r}) = \left(1 + \frac{m}{2\tilde{r}}\right)^2 \tag{6}$$

giving the metric

$$ds^2 = -\left(\frac{\tilde{r} - m/2}{\tilde{r} + m/2}\right)^2 dt^2 + \left(1 + \frac{m}{2\tilde{r}}\right)^4 (d\tilde{r}^2 + \tilde{r}^2 d\Omega^2). \tag{7}$$

2 Derive the red-shift formula for radial photons moving in the metric

$$g = -\left(1 - \frac{a}{r}\right)^3 dt^2 + \frac{dr^2}{\left(1 - \frac{a}{r}\right)^3} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \,. \tag{8}$$

By analyzing the geodesic equation at large distances or otherwise, derive the force acting on test bodies moving in this gravitational field for large r and for small velocities with respect to the static observers.