1 Isotropic coordinates for the Schwarzschild metric. We wish to find coordinates \((t, \tilde{r}, \theta, \phi)\), where \(\tilde{r} = \tilde{r}(r)\), for the Schwarzschild metric in terms of which the spatial metric is conformally flat. In particular, we wish to find functions \(\tilde{r}(r)\), \(A(\tilde{r})\) and \(B(\tilde{r})\) such that

\[
ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2d\Omega^2
\]

\[= -A(\tilde{r})^2dt^2 + B(\tilde{r})^2(d\tilde{r}^2 + \tilde{r}^2d\Omega^2). \tag{1}\]

Q1. Deduce that we require

\[
A(\tilde{r})^2 = 1 - \frac{2m}{r},
\]

\[
B(\tilde{r})^2 \left(\frac{d\tilde{r}}{dr}\right)^2 = \frac{r}{r - 2m},
\]

\[
B(\tilde{r})^2\tilde{r}^2 = r^2. \tag{2}\]

Q2. Assuming that \(d\tilde{r}/dr > 0\) and \(\tilde{r} > 0\), show that \(\tilde{r}\) satisfies the relation

\[
\frac{1}{\tilde{r}} \frac{d\tilde{r}}{dr} = \frac{1}{\sqrt{r(r - 2m)}}. \tag{3}\]

Integrating with respect to \(r\) from \(r = 2m\) and letting \(\tilde{r}(2m) = m/2\), show that

\[
\tilde{r}(r) = \frac{m}{2} \left(\frac{r - m}{m} + \sqrt{\left(\frac{r - m}{m}\right)^2 - 1}\right) \tag{4}\]

and

\[
r = \frac{1}{\tilde{r}} \left(\tilde{r} + \frac{m}{2}\right)^2. \tag{5}\]

Q3. Show that

\[
A(\tilde{r})^2 = \left(\frac{\tilde{r} + m/2}{2}\right)^2, \quad B(\tilde{r}) = \left(1 + \frac{m}{2\tilde{r}}\right)^2 \tag{6}\]

giving the metric

\[
ds^2 = -\left(\frac{\tilde{r} - m/2}{\tilde{r} + m/2}\right)^2 dt^2 + \left(1 + \frac{m}{2\tilde{r}}\right)^4 (d\tilde{r}^2 + \tilde{r}^2d\Omega^2). \tag{7}\]

2 Derive the red-shift formula for radial photons moving in the metric

\[
g = -\left(1 - \frac{a}{r}\right)^3 dt^2 + \frac{dr^2}{\left(1 - \frac{a}{r}\right)^3} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \tag{8}\]

By analyzing the geodesic equation at large distances or otherwise, derive the force acting on test bodies moving in this gravitational field for large \(r\) and for small velocities with respect to the static observers.