1 A non-physical black hole metric. Let a be a strictly positive constant, and for $r > a$ let g be a Lorentzian metric of the form

$$g = -\left(1 - \frac{a}{r}\right)^3 dt^2 + \frac{dr^2}{\left(1 - \frac{a}{r}\right)^3} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \, . \tag{1}$$

Q1. Proceeding in a way analogous to the analysis of the Schwarzschild metric, replace t by a new coordinate v so that the metric, in the new coordinates, can be smoothly extended from the original manifold $\{t \in \mathbb{R}\} \times \{r > a\} \times S^2$ to a Lorentzian metric on a new manifold $\{v \in \mathbb{R}\} \times \{r > 0\} \times S^2$.

Q2. Show that, with an appropriate choice of the coordinate v, the set $\{r < a\}$ in the extended manifold is a black hole region.

Q3. Find the four-acceleration of stationary observers for this metric.

Q4. How would you show that $\{r = 0\}$ is a singular set for the metric (1)? For the ambitious: use an algebraic manipulation program to carry this out.

2 Radial geodesics in Schwarzschild: Consider the geodesic equations for a massive particle initially at rest in the Schwarzschild metric. The tangent vector $\mathbf{U} = \frac{d\gamma}{ds}$ therefore satisfies $U^a \nabla_a U^b = 0$ and $g_{ab} U^a U^b = -1$ with the initial conditions that $U^r(s = 0) = U^\theta(s = 0) = U^\phi(s = 0) = 0$ and $U^t(s = 0) > 0$.

a). Show that we have

$$U^t(0) \equiv \frac{dt}{ds}(s = 0) = \frac{1}{\sqrt{1 - \frac{2m}{r(0)}}}.$$

b). Derive the part of the geodesic equations for $\frac{d}{ds}U^\theta$, $\frac{d}{ds}U^\phi$. [Hint: Repeat the argument presented in the lecture.] Show that the initial conditions $U^\theta(0) = U^\phi(0) = 0$ imply that $U^\theta(s) = U^\phi(s) = 0$ for all s. Therefore, the geodesics are radial (i.e. the only spatial motion is in the radial direction).

c). Using a formula derived in the lecture (the derivation of which you should reproduce), show that

$$\frac{dt}{ds} = \frac{\sqrt{1 - \frac{2m}{r(0)}}}{1 - \frac{2m}{r(s)}}$$

d). From the result of part c), and the fact that $g_{ab} U^a U^b = -1$, deduce that $r(s)$ satisfies the differential equation

$$\frac{dr}{ds} = -\sqrt{\frac{2m}{r(s)} - \frac{2m}{r(0)}}.$$

[Note: At one point, you will need to take a square root, which involves a choice of sign. In order that $\frac{2m}{r(s)} - \frac{2m}{r(0)}$ is positive, you need $r(s) \leq r(0)$, which tells you that you should choose the negative root.] Integrating this equation, show that $r(s)$ is implicitly given by the equation

$$s = -\frac{r(0)^{3/2}}{\sqrt{2m}} \left[\cos^{-1} \sqrt{\frac{r}{r_0}} - \sqrt{\frac{r}{r_0}} \left(1 - \left(\frac{r}{r^0} \right) \right)^{1/2} \right].$$

[Hint: When doing the integration, you may find it useful to use the substitution $r = r_0 \cos^2 x$.]