1 Some timelike geodesics in Schwarzschild. Let \( V^2 = 1 - 2m/r \), and consider the Schwarzschild metric:

\[
g = -(1 - \frac{2m}{r})dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2d\Omega^2, \tag{1}
\]

\[
t \in \mathbb{R}, \quad r \neq 2m, 0, \tag{2}
\]

Recall, from a previous problem sheet, that for geodesics in the Schwarzschild metric lying in the equatorial plane \( \theta = \pi/2 \) we have the following constants of motion (you are strongly encouraged to check that you can reproduce the result, but this will not be done again in class):

\[
\frac{d}{ds}\left(\frac{V^2}{2} \frac{dt}{ds}\right) = 0 \quad \Rightarrow \quad \frac{dt}{ds} = \frac{E}{1 - \frac{2m}{r}}. \tag{3}
\]

\[
\frac{d}{ds}\left(\frac{r^2}{2} \frac{d\varphi}{ds}\right) = 0 \quad \Rightarrow \quad \frac{d\varphi}{ds} = \frac{J}{r^2}. \tag{4}
\]

\[
\frac{V^2}{E^2V^{-2}} \left(\frac{dt}{ds}\right)^2 - \frac{\left(\frac{dr}{ds}\right)^2}{\frac{2}{r^2}} - \frac{\left(\frac{d\varphi}{ds}\right)^2}{\frac{2}{r^2}} = \lambda \in \{0, \pm1\}. \tag{5}
\]

Consider a timelike geodesic in the Schwarzschild metric parameterized by proper time, thus \( \lambda = 1 \), and lying in the equatorial plane \( \theta = \pi/2 \).

Q1. Verify that

\[
\frac{E^2 - r^2}{1 - \frac{2m}{r}} - \frac{J^2}{r^2} = 1.
\]

Q2. Deduce that if \( E = 1 \) and \( J = 4m \) then

\[
\frac{\sqrt{r} - 2 \sqrt{m}}{\sqrt{r} + 2 \sqrt{m}} = Ae^{\epsilon \varphi/\sqrt{2}},
\]

where \( \epsilon = \pm 1 \) and \( A \) is a constant. Describe the orbit that starts at \( \varphi = 0 \) in each of the cases (i) \( A = 0 \), (ii) \( A = 1 \), \( \epsilon = -1 \), (iii) \( r(0) = 3m \), \( \epsilon = -1 \).

Q3: Consider the case \( E = 1 \), \( J = 0 \) and deduce a relation for \( r = r(s) \).

2 Null Geodesics in Schwarzschild Let \( \gamma \) be an affinely parameterized null geodesic, thus \( \lambda = 0 \), in the Schwarzschild metric lying in the equatorial plane \( \theta = \pi/2 \). Assume that \( J \neq 0 \), hence we can make a change of parameter \( \varphi \mapsto s(\varphi) \) using the implicit equation

\[
\frac{d\varphi}{ds} = \frac{J}{r^2} \quad \Rightarrow \quad \frac{ds}{d\varphi} = \frac{r^2}{J}.
\]
Set
\[ u(\varphi) = \frac{m}{r(s(\varphi))}, \quad p(\varphi) = \frac{du(\varphi)}{d\varphi}. \]

Q1. Show that along $\gamma$, for $u \neq \frac{1}{2}$, we have
\[ p^2 = 2u^3 - u^2 + \alpha^2. \] (6)

Q2. For the case $\alpha = 0$ show that we obtain an autonomous two-dimensional system
\[ \frac{du}{d\varphi} = p, \quad \frac{dp}{d\varphi} = 3u^2 - u. \]

Recall that critical points of a dynamical system $d\vec{x}/d\varphi = \vec{Y}$ (here $\vec{x} = (u, p)$) are defined as points where $\vec{Y}$ vanishes. Find the critical points. Can you sketch the trajectories in the $(u, p)$ phase-plane?

Q3. For the case $\alpha = 0$ and $u > 1/2$ show that the geodesic has an equation of the form
\[ r = 2m \cos^2 \left( \frac{\varphi - \varphi_0}{2} \right), \] (7)
with $t = t(\varphi)$ that you should determine.