1 Some tensor manipulations

i. Show that for all functions f

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f$$

if and only if ∇ is torsion-free.

ii. Recall that ∇f is defined as the vector field $g^{ab}\partial_a f \partial_b$. Explain what $df(\nabla f)$ and $\nabla f(f)$ mean, and prove the equalities

$$g(\nabla f, \nabla f) = g^{ab}\partial_a f \partial_b f = df(\nabla f) = \nabla f(f) \ .$$

Conclude that if $f = x^\alpha$ is a coordinate, you can determine whether ∇f is timelike, spacelike, or null, by looking at a component of the inverse metric tensor (which?).

2 A gravitational wave

Let η_{ab} and η^{ab} be the covariant and contravariant metric tensors on Minkowski space $\mathcal{M}$, with standard Lorentzian coordinates x^a so that

$$(\eta_{ab}) = -\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{1}$$

Let n_a be a constant covector on $\mathcal{M}$ satisfying $\eta^{ab}n_a n_b = 0$. Define a new metric on $\mathcal{M}$ by

$$g_{ab} = \eta_{ab} + f n_a n_b \ ,$$

where f is a function on $\mathcal{M}$ such that $\eta^{ab}n_a\partial_b f = 0$, where $\partial_a = \partial/\partial x^a$.

i. Show that the connection derived from g_{ab} is given by

$$\Gamma^a{}_{bc} = \tfrac{1}{2}\eta^{ad}\left(n_d n_b \partial_c f + n_d n_c \partial_b f - n_b n_c \partial_d f\right) \ .$$

[*Hint: look for g^{ab} of the form $\eta^{ab} + hn^a n^b$.*]

ii. The Ricci tensor is defined as

$$R_{ac} = \partial_d \Gamma^d{}_{ac} - \partial_a \Gamma^d{}_{dc} + \Gamma^d{}_{de}\Gamma^e{}_{ac} - \Gamma^d{}_{ae}\Gamma^e{}_{dc} \ . \tag{2}$$

Show that

$$R_{ab} = -\tfrac{1}{2} n_a n_b \eta^{cd}\partial_c \partial_d f \ , \tag{3}$$

[*Hint: Check that $\Gamma^d{}_{de}$ vanishes. Show, next, that any contraction of n^a with the Christoffel symbols vanishes, and conclude that the last term in (2) gives no contribution either.*]

iii. Show that Einstein's vacuum field equations,

$$R_{ab} = 0 \ ,$$

have solutions as above with $f = \alpha \sin(k_a x^a)$, with $\alpha \in \mathbb{R}$, and with k_a having constant components in the coordinate system where η_{ab} takes the form (1), provided that k_a satisfies $\eta^{ab} k_a k_b = \eta^{ab} k_a n_b = 0$. Deduce that such a k_a is proportional to n_a.