1 Let

$$g_{\mu\nu} = \eta_{\mu\nu} + \Re\left(A_{\mu\nu}\exp(ik_\alpha x^\alpha)\right) , \quad A_{\mu\nu} \in \mathbb{C} ,$$

be the metric of a linearized plane wave in the gauge $A_{\mu\nu}k^\mu = 0 = A_{\mu\nu}u^\mu = A_\mu{}^\mu$, where $k^\mu\partial_\mu = \omega(\partial_t + \partial_z)$, $u^\mu\partial_\mu = \partial_t$. Introducing the variables $u = t - z$, $v = t + z$, find the geodesics, neglecting terms which are quadratic in the $A_{\mu\nu}$'s when helpful.

2 Consider a pointlike body of mass m_0 and angular momentum J_N moving in an elliptic, Kepler orbit of eccentricity e in the field of a central mass of mass m. The density distribution of the central mass is spherically symmetric and time-independent. Assume that, in Cartesian coordinates, this motion takes place in the $z = 0$ plane. Show that the *time-dependent part* of the quadrupole moment of the system takes the form

$$q_{ij} = 3m_0 r(\varphi(t))^2 \begin{pmatrix} \cos^2\varphi(t) & \sin\varphi(t)\cos\varphi(t) & 0 \\ \sin\varphi(t)\cos\varphi(t) & \sin^2\varphi(t) & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Using the explicit form of $r(\varphi(t))$ derived in lectures, derive an expression for the linearised metric perturbation $\overline{h}_{xx}$, to first order in the eccentricity e.

3 Consider the metric

$$g = g_{\text{Schw}} - \frac{4J\sin^2\theta}{r} dt\, d\varphi ,$$

where g_{Schw} denotes the Schwarzschild metric.

i. Show that are geodesics for the metric g of the form $\gamma(\tau) = (t(\tau), r(\tau), \theta = 0)$.

ii. Show that if $s^r = 0$ at some point then it is zero along the whole such geodesic γ.

iii. Write-down explicitly the gyroscope equation on γ assuming that $s^r = 0$. (Can you solve it?)