1 Assuming a circular orbit, calculate m/r, J_N/r^2, m/J_N^2 for a) the orbit of the earth around the sun, and for b) the orbit of Mercury around the sun.

Using the formula from the lecture (you might wish, though, to verify that you can reproduce its derivation), calculate the shift of the periastron for Mars and the earth in orbit around the Sun. Repeat the calculation for an exoplanet with the mass of Jupiter on a circular orbit around a star of 10 solar masses with a period of 1 day.

2 Timelike geodesics in Schwarzschild. Let γ be a timelike geodesic for the Schwarzschild metric parameterized by proper time and lying in the equatorial plane $\theta = \pi/2$.

Q1. Show that

$$\frac{E^2 - \dot{r}^2}{1 - \frac{2m}{r}} - \frac{J^2}{r^2} = 1 \ .$$

Q2. Deduce that if $E = 1$ and $J = 4m$ then

$$\frac{\sqrt{r} - 2\sqrt{m}}{\sqrt{r} + 2\sqrt{m}} = Ae^{\epsilon\varphi/\sqrt{2}} \ ,$$

where $\epsilon = \pm 1$ and A is a constant. Describe the orbit that starts at $\varphi = 0$ in each of the cases (i) $A = 0$, (ii) $A = 1$, $\epsilon = -1$, (iii) $r(0) = 3m$, $\epsilon = -1$.

3 Derive the red-shift formula for radial photons moving in the metric

$$g = -\left(1 - \frac{a}{r}\right)^3 dt^2 + \frac{dr^2}{\left(1 - \frac{a}{r}\right)^3} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \ . \qquad (1)$$

By analyzing the geodesic equation at large distances or otherwise, derive the force acting on test bodies moving in this gravitational field for large r and for small velocities with respect to the static observers.