1 Radial geodesics in Schwarzschild: Consider the geodesic equations for a massive particle initially at rest in the Schwarzschild metric. The tangent vector $\mathbf{U} = \frac{d\gamma}{ds}$ therefore satisfies $U^a \nabla_a U^b = 0$ and $g_{ab} U^a U^b = -1$ with the initial conditions that $U^r(s=0) = U^\theta(s=0) = U^\phi(s=0) = 0$ and $U^t(s=0) > 0$.

a). Show that we have

$$U^t(0) \equiv \frac{dt}{ds}(s=0) = \frac{1}{\sqrt{1 - \frac{2m}{r(0)}}}.$$

b). Derive the part of the geodesic equations for $\frac{d}{ds}U^\theta, \frac{d}{ds}U^\phi$. [Hint: Repeat the argument presented in the lecture.] Show that the initial conditions $U^\theta(0) = U^\phi(0) = 0$ imply that $U^\theta(s) = U^\phi(s) = 0$ for all $s$. Therefore, the geodesics are radial (i.e. the only spatial motion is in the radial direction).

c). Using a formula derived in the lecture (the derivation of which you should reproduce), show that

$$\frac{dt}{ds} = \frac{\sqrt{1 - \frac{2m}{r(0)}}}{1 - \frac{2m}{r(s)}}$$

d). From the result of part c), and the fact that $g_{ab} U^a U^b = -1$, deduce that $r(s)$ satisfies the differential equation

$$\frac{dr}{ds} = -\sqrt{\frac{2m}{r(s)} - \frac{2m}{r(0)}}.$$

[Note: At one point, you will need to take a square root, which involves a choice of sign. In order that $\frac{2m}{r(s)} - \frac{2m}{r(0)}$ is positive, you need $r(s) \le r(0)$, which tells you that you should choose the negative root.] Integrating this equation, show that $r(s)$ is implicitly given by the equation

$$s = -\frac{r(0)^{3/2}}{\sqrt{2m}} \left[ \cos^{-1} \sqrt{\frac{r}{r_0}} - \sqrt{\frac{r}{r_0}} \left( 1 - \left( \frac{r}{r^0} \right)^2 \right)^{1/2} \right].$$

[Hint: When doing the integration, you may find it useful to use the substitution $r = r_0 \cos^2 x$.]

2 Null Geodesics in Schwarzschild Let $\gamma$ be an affinely parameterized null geodesic for the Schwarzschild metric with $J \neq 0$ lying in the equatorial plane $\theta = \pi/2$.

Q1. Show that along $\gamma$, both for $u > 1/2$ and for $u < 1/2$, we have

$$p^2 = 2u^3 - u^2 + \alpha^2 , \tag{1}$$

where $\alpha$ is a constant and

$$u = \frac{m}{r}, \quad p = \frac{du}{d\varphi} .$$

Q2. For the case $\alpha = 0$ show that we obtain an autonomous two-dimensional system

$$\frac{du}{d\phi} = p \ , \quad \frac{dp}{d\phi} = 3u^2 - u \ .$$

Find and classify the critical points of this system of ordinary differential equations.[1] Sketch the trajectories in the $(u, p)$ phase-plane.

Q3. For the case $\alpha = 0$ and $u > 1/2$ show that the geodesic has an equation of the form

$$r = 2m \cos^2\left(\frac{\varphi - \varphi_0}{2}\right) \ , \tag{2}$$

with $t = t(\varphi)$ that you should determine.

[1] Recall that a critical point of a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a point at which $\mathbf{f} = 0$. Critical points are classified by analysing the linearisation around the critical point. See `http://mathworld.wolfram.com/FixedPoint.html` for a concise summary and, for example, the beginning of Chapter 4 of `http://math.columbusstate.edu/ejionascu/papers/diffeqbook.pdf` for details.