1 A non-physical black hole metric. Let a be a strictly positive constant, and for $r > a$ let g be a Lorentzian metric of the form

$$g = -\left(1 - \frac{a}{r}\right)^3 dt^2 + \frac{dr^2}{\left(1 - \frac{a}{r}\right)^3} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \, . \tag{1}$$

Q1. Proceeding in a way analogous to the analysis of the Schwarzschild metric, replace t by a new coordinate v so that the metric, in the new coordinates, can be smoothly extended from the original manifold $\{t \in \mathbb{R}\} \times \{r > a\} \times S^2$ to a Lorentzian metric on a new manifold $\{v \in \mathbb{R}\} \times \{r > 0\} \times S^2$.

Q2. Show that, with an appropriate choice of the coordinate v, the set $\{r < a\}$ in the extended manifold is a black hole region.

Q3. Find the four-acceleration of stationary observers for this metric.

Q4. How would you show that $\{r = 0\}$ is a singular set for the metric (1)? For the ambitious: use an algebraic manipulation program to carry this out.

2 Isotropic coordinates for the Schwarzschild metric.

We wish to find coordinates $(t, \tilde{r}, \theta, \phi)$, where $\tilde{r} = \tilde{r}(r)$, for the Schwarzschild metric in terms of which the spatial metric is conformally flat. In particular, we wish to find functions $\tilde{r}(r)$ and functions $A(\tilde{r})$, $B(\tilde{r})$ such that

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \\ &= -A(\tilde{r})^2 dt^2 + B(\tilde{r})^2 \left[d\tilde{r}^2 + \tilde{r}^2 d\Omega^2 \right]. \end{aligned}$$

Q1. Deduce that we require

$$\begin{aligned} A(\tilde{r})^2 &= 1 - \frac{2m}{r}, \\ B(\tilde{r})^2 \left(\frac{d\tilde{r}}{dr}\right)^2 &= \frac{r}{r - 2m}, \\ B(\tilde{r})^2 \tilde{r}^2 &= r^2. \end{aligned}$$

Q2. Assuming that $\frac{d\tilde{r}}{dr} > 0$ and $\tilde{r} > 0$, show that $\tilde{r}$ satisfies the relation

$$\frac{1}{\tilde{r}} \frac{d\tilde{r}}{dr} = \frac{1}{\sqrt{r\,(r - 2m)}}.$$

Integrating with respect to r from $r = 2m$ and letting $\tilde{r}(2m) = \frac{m}{2}$, show that

$$\tilde{r}(r) = \frac{m}{2} \cdot \left[\frac{r - m}{m} + \sqrt{\left(\frac{r - m}{m}\right)^2 - 1} \right]$$

and, hence,

$$r = \frac{1}{\widetilde{r}}\left(\widetilde{r} + \frac{m}{2}\right)^2 .$$

Q3. Show that

$$A(\widetilde{r})^2 = \left(\frac{\widetilde{r} - \frac{m}{2}}{\widetilde{r} + \frac{m}{2}}\right)^2 , \qquad B(\widetilde{r}) = \left(1 + \frac{m}{2\widetilde{r}}\right)^2 ,$$

giving the metric

$$ds^2 = -\left(\frac{\widetilde{r} - \frac{m}{2}}{\widetilde{r} + \frac{m}{2}}\right)^2 dt^2 + \left(1 + \frac{m}{2\widetilde{r}}\right)^4 \left[d\widetilde{r}^2 + \widetilde{r}^2 d\Omega^2\right].$$