

## Übungen zur Vorlesung Relativitätstheorie und Kosmologie II: Problem Sheet 1

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### 1 Some tensor manipulations

- i. Show that for all functions  $f$

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f$$

if and only if  $\nabla$  is torsion-free.

- ii. Recall that  $\nabla f$  is defined as the vector field  $g^{ab} \partial_a f \partial_b$ . Explain what  $df(\nabla f)$  and  $\nabla f(f)$  mean, and prove the equalities

$$g(\nabla f, \nabla f) = g^{ab} \partial_a f \partial_b f = df(\nabla f) = \nabla f(f) .$$

Conclude that if  $f = x^\alpha$  is a coordinate, you can determine whether  $\nabla f$  is timelike, spacelike, or null, by looking at a component of the inverse metric tensor (which?).

### 2 A gravitational wave

Let  $\eta_{ab}$  and  $\eta^{ab}$  be the covariant and contravariant metric tensors on Minkowski space  $\mathcal{M}$ , with standard Lorentzian coordinates  $x^a$  so that

$$(\eta_{ab}) = - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \quad (1)$$

Let  $n_a$  be a constant covector on  $\mathcal{M}$  satisfying  $\eta^{ab} n_a n_b = 0$ . Define a new metric on  $\mathcal{M}$  by

$$g_{ab} = \eta_{ab} + f n_a n_b ,$$

where  $f$  is a function on  $\mathcal{M}$  such that  $\eta^{ab} n_a \partial_b f = 0$ , where  $\partial_a = \partial/\partial x^a$ .

- i. Show that the connection derived from  $g_{ab}$  is given by

$$\Gamma^a_{bc} = \frac{1}{2} \eta^{ad} (n_d n_b \partial_c f + n_d n_c \partial_b f - n_b n_c \partial_d f) .$$

[Hint: look for  $g^{ab}$  of the form  $\eta^{ab} + h n^a n^b$ .]

- ii. The Ricci tensor is defined as

$$R_{ac} = \partial_d \Gamma^d_{ac} - \partial_a \Gamma^d_{dc} - \Gamma^d_{de} \Gamma^e_{ac} + \Gamma^d_{ae} \Gamma^e_{dc} . \quad (2)$$

Show that

$$R_{ab} = \frac{1}{2} n_a n_b \eta^{cd} \partial_c \partial_d f , \quad (3)$$

[Hint: Check that  $\Gamma^d_{de}$  vanishes. Show, next, that any contraction of  $n^a$  with the Christoffel symbols vanishes, and conclude that the last term in (2) gives no contribution either.]

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- iii. Show that Einstein's vacuum field equations,

$$R_{ab} = 0 ,$$

have solutions as above with  $f = \alpha \sin(k_a x^a)$ , with  $\alpha \in \mathbb{R}$ , and with  $k_a$  having constant components in the coordinate system where  $\eta_{ab}$  takes the form (1), provided that  $k_a$  satisfies  $\eta^{ab} k_a k_b = \eta^{ab} k_a n_b = 0$ . Deduce that such a  $k_a$  is proportional to  $n_a$ .