1 QI. The Lane-Emden equation

Recall from lectures that, for a Newtonian static fluid, $p$ and $\rho$ satisfy the equation

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dp}{d\rho} \right) = -4\pi \rho.$$

Assume that $\rho = \rho_c \theta(r)^n$ and $p = p_c \theta(r)^{n+1}$, where $n, \rho_c, p_c$ are constants, and $\theta$ is a function to be determined. Check that

$$p = K \rho^{\frac{n+1}{n}},$$

for a constant $K$ that you will determine. Show that the function $\theta$ satisfies the differential equation

$$\frac{K(n+1)}{4\pi \rho_c^{\frac{1}{n} - 1}} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\theta}{dr} \right) = -\theta^n.$$

Let $r_n > 0$ be defined by

$$r_n^2 = \frac{K(n+1)}{4\pi \rho_c^{\frac{1}{n} - 1}},$$

and define a new variable $\xi$ by $r = r_n \xi$. Deduce that

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n.$$

Solve this equation with boundary conditions $\theta(0) = 1, \theta'(0) = 0$ in the case $n = 0$. In the case $n = 1$, let $\theta_1(\xi) = \theta(\xi)/\xi$, and solve for $\theta_1(\xi)$ with the same boundary conditions. Finally, verify that the function

$$\theta_5(\xi) = \frac{1}{\left(1 + \frac{1}{3} \xi^2\right)^{1/2}}$$

is a solution in the case $n = 5$.

2 Given a vector field $X$ on a manifold, the flow of $X$ is the solution $x^a(t)$ of the ordinary differential equation $\frac{dx^a}{dt} = X^a(x(t))$. Given arbitrary initial conditions, find the flow of the following vector fields:

i. $X = x \partial_x + y \partial_y$ on $\mathbb{R}^2$

ii. $X = x \partial_y - y \partial_x$ on $\mathbb{R}^3$

iii. Let $M = \{a = (a_{ij}), \ a_{ij} \in \mathbb{R}\}$ be the set of all $n \times n$ matrices, and let $X(a) = a_{ij} \frac{\partial}{\partial a_{ij}}$. 
3. Given a one-form, $\alpha$, we define its exterior derivative, $d\alpha$, as

$$d\alpha_{ab} = \partial_a \alpha_b - \partial_b \alpha_a.$$ 

Show that $d\alpha$ is a tensor.

Show that, acting on arbitrary vector fields $X$, $Y$, we have 

$$d\alpha(X, Y) = \mathcal{L}_X (\alpha(Y)) - \mathcal{L}_Y (\alpha(X)) - \alpha([X, Y]).$$

(Note that this allows us to define the exterior derivative of a one-form in terms of the Lie derivative.) Also, show that for arbitrary vector fields $X$, $Y$, we have 

$$\mathcal{L}_X (\alpha) (Y) = d\alpha(X, Y) + Y(\alpha(X)).$$

Recall that if $f$ is a function, then $df = \partial_i f dx^i$. Show that $d(df) = 0$, and 

$$\mathcal{L}_X (df) = d(\mathcal{L}_X f).$$