1 QI. The Lane-Emden equation

Recall from lectures that, for a Newtonian static fluid, p and ρ satisfy the equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi\rho.$$

Assume that $\rho = \rho_c\theta(r)^n$ and $p = p_c\theta(r)^{n+1}$, where n, ρ_c, p_c are constants, and θ is a function to be determined. Check that

$$p = K\rho^{\frac{n+1}{n}},$$

for a constant K that you will determine. Show that the function θ satisfies the differential equation

$$\frac{K(n+1)}{4\pi}\rho_c^{\frac{1}{n}-1}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n.$$

Let $r_n > 0$ be defined by

$$r_n^2 = \frac{K(n+1)}{4\pi}\rho_c^{\frac{1}{n}-1},$$

and define a new variable ξ by $r = r_n\xi$. Deduce that

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n.$$

Solve this equation with boundary conditions $\theta(0) = 1$, $\theta'(0) = 0$ in the case $n = 0$. In the case $n = 1$, let $\theta_1(\xi) = \theta(\xi)/\xi$, and solve for $\theta_1(\xi)$ with the same boundary conditions. Finally, verify that the function

$$\theta_5(\xi) = \frac{1}{\left(1 + \frac{1}{3}\xi^2\right)^{1/2}}$$

is a solution in the case $n = 5$.

2 Given a vector field X on a manifold, the *flow* of X is the solution $x^a(t)$ of the ordinary differential equation $\frac{dx^a}{dt} = X^a(x(t))$. Given arbitrary initial conditions, find the flow of the following vector fields:

i. $X = x\partial_x + y\partial_y$ on $\mathbb{R}^2$

ii. $X = x\partial_y - y\partial_x$ on $\mathbb{R}^3$

iii. Let $M = \{a = (a_{ij})\ ,\ a_{ij} \in \mathbb{R}\}$ be the set of all $n \times n$ matrices, and let $X(a) = a_{ij}\frac{\partial}{\partial a_{ij}}$.

3 Given a one-form, α, we define its exterior derivative, $d\alpha$, as

$$d\alpha_{ab} = \partial_a \alpha_b - \partial_b \alpha_a.$$

Show that $d\alpha$ is a tensor.

Show that, acting on arbitrary vector fields X, Y, we have

$$d\alpha(X, Y) = \mathcal{L}_X (\alpha(Y)) - \mathcal{L}_Y (\alpha(X)) - \alpha([X, Y]).$$

(Note that this allows us to define the exterior derivative of a one-form in terms of the Lie derivative.) Also, show that for arbitrary vector fields X, Y, we have

$$(\mathcal{L}_X \alpha)(Y) = d\alpha(X, Y) + Y(\alpha(X)) .$$

Recall that if f is a function, then $df = \partial_i f dx^i$. Show that $d(df) = 0$, and

$$\mathcal{L}_X(df) = d(\mathcal{L}_X f) .$$