QI. Lie derivative

Given a vector field X, the *Lie derivative* $\mathcal{L}_X$ is an operation on tensors, defined as follows:

i. For a function f, one sets $\mathcal{L}_X f := X(f)$.

ii. For a vector field Y, one sets $\mathcal{L}_X Y := [X, Y]$, the Lie bracket.

iii. For a one form α, $\mathcal{L}_X\alpha$ is defined by imposing the Leibniz rule written backwards:

$$(\mathcal{L}_X\alpha)(Y) := \mathcal{L}_X(\alpha(Y)) - \alpha(\mathcal{L}_X Y) \ .$$

Q I.1: a) Why is this the same as the Leibniz rule? *[Hint: write this equation using indices.]* b) Check that $\mathcal{L}_X\alpha$ is a tensor. *[Hint: check that the right-hand-side is linear under multiplication of Y by a function.]*

Q I.2: Show that

$$(\mathcal{L}_X\alpha)_a = X^b\partial_b\alpha_a + \alpha_b\partial_a X^b \ .$$

iv. For tensor products the Lie derivative is defined again by imposing linearity under addition together with the Leibniz rule:

$$\mathcal{L}_X(\alpha\otimes\beta) = (\mathcal{L}_X\alpha)\otimes\beta + \alpha\otimes\mathcal{L}_X\beta \ .$$

Since a general tensor A is sum of tensor products,

$$A = A^{a_1\ldots a_p}{}_{b_1\ldots b_q}\partial_{a_1}\otimes\ldots\partial_{a_p}\otimes dx^{b_1}\otimes\ldots\otimes dx^{a_p} \ ,$$

requiring linearity with respect to addition of tensors gives thus a definition of Lie derivative for any tensor.

Q I.3: Show that

$$\mathcal{L}_X T^a{}_b = X^c\partial_c T^a{}_b - T^c{}_b\partial_c X^a + T^a{}_c\partial_b X^c \ ,$$

Q I.4: Can you see the general formula for the Lie derivative $\mathcal{L}_X A^{a_1\ldots a_p}{}_{b_1\ldots b_q}$?