QI. Lie derivative

Given a vector field $X$, the *Lie derivative* $\mathcal{L}_X$ is an operation on tensors, defined as follows:

i. For a function $f$, one sets $\mathcal{L}_X f := X(f)$.

ii. For a vector field $Y$, one sets $\mathcal{L}_X Y := [X, Y]$, the Lie bracket.

iii. For a one form $\alpha$, $\mathcal{L}_X \alpha$ is defined by imposing the Leibniz rule written backwards:

$$ (\mathcal{L}_X \alpha)(Y) := \mathcal{L}_X(\alpha(Y)) - \alpha(\mathcal{L}_X Y) . $$

Q I.1: a) Why is this the same as the Leibniz rule? *[Hint: write this equation using indices.]*

b) Check that $\mathcal{L}_X \alpha$ is a tensor. *[Hint: check that the right-hand-side is linear under multiplication of $Y$ by a function.]*

Q I.2: Show that

$$ (\mathcal{L}_X \alpha)_a = X^b \partial_b \alpha_a + \alpha_b \partial_a X^b . $$

iv. For tensor products the Lie derivative is defined again by imposing linearity under addition together with the Leibniz rule:

$$ \mathcal{L}_X (\alpha \otimes \beta) = (\mathcal{L}_X \alpha) \otimes \beta + \alpha \otimes (\mathcal{L}_X \beta) . $$

Since a general tensor $A$ is sum of tensor products,

$$ A = A^{a_1 \ldots a_p}{}_{b_1 \ldots b_q} \partial_{a_1} \otimes \ldots \partial_{a_p} \otimes dx^{b_1} \otimes \ldots \otimes dx^{a_p} , $$

requiring linearity with respect to addition of tensors gives thus a definition of Lie derivative for any tensor.

Q I.3: Show that

$$ \mathcal{L}_X T^{a}{}_{b} = X^c \partial_c T^{a}{}_{b} - T^{c}{}_{b} \partial_c X^a + T^{a}{}_{c} \partial_b X^c , $$

Q I.4: Can you see the general formula for the Lie derivative $\mathcal{L}_X A^{a_1 \ldots a_p}{}_{b_1 \ldots b_q}$?