Übungen zur Vorlesung Relativitätstheorie und Kosmologie II: Revision Problem Sheet

1 Lie bracket

Recall that vector fields can be identified with homogeneous linear first order partial differential operators $X = X^a\partial_a$ acting on functions as $X(f) = X^a\partial_a f$.

The Lie-bracket $[X, Y]$ of two vector fields X and Y is defined as

$$[X, Y](f) = X(Y(f)) - Y(X(f)) .$$

Show that $[X, Y]$ also is a vector field, i.e. a homogeneous linear first order differential operator, with components

$$[X, Y]^a = X^b\partial_b Y^a - Y^b\partial_b X^a . \tag{1}$$

Check, by a direct coordinate calculation, that the right-hand-side of (1) transforms as a vector field under changes of coordinates.

Prove the *Jacobi identity*:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 .$$

2 Some geodesics

Recall that the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = \frac{\partial L}{\partial x^a} \tag{2}$$

associated with the Lagrange function

$$L(x^c, \dot{x}^c) = \frac{1}{2} g_{ab}\dot{x}^a\dot{x}^b \tag{3}$$

can be written as

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0 , \tag{4}$$

which is the *geodesic equation*.

Using the above variational principle for geodesics, write down the geodesic equations, and give the obvious constants of motion, for

a) the *Schwarzschild* metric

$$g_m = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{1}{1 - \frac{2m}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

(you might wish to do the calculation for a metric of the form

$$-e^{2f(r)}dt^2 + e^{-2f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) ,$$

and specialize the result to Schwarzschild at the end, as many other important metrics are of this form, so the result might be useful later). Show that a geodesic initially tangent to the equatorial plane always remains in it.

Use (4) to calculate the Christoffel symbols for the Schwarzschild metric.

b) The following *pp-wave* metric

$$g = dx^2 + dy^2 - 2du\,dv + H(u,x)du^2\ .$$

c) The "post-Newtonian" metric

$$g_{00} = -(1 - \frac{2GM}{r})\ , \quad g_{0i} = 0\ , \quad g_{ij} = \Big(1 + \frac{2GM}{r}\Big)\delta_{ij}\ ,$$

with $i, j \in \{1, 2, 3\}$. (This is the Newtonian approximation, for $GM/r \ll 1$, of the metric tensor of a spherically symmetric body of mass M.) In this calculation neglect all terms quadratic in GM.