5.3 A Toy model: the scalar field in Minkowski space-time

We want to derive a formula for the amount of energy radiated by a scalar field ϕ satisfying a sourceless wave equation in Minkowski space-time :

$$\Box_\eta \phi = 0 \ .$$

We will use the energy-momentum tensor

$$\begin{aligned} T_{\mu\nu} &= \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}\nabla^\alpha\phi\nabla_\alpha\phi\eta_{\mu\nu} \\ &= \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\eta_{\mu\nu} \ . \end{aligned} \qquad (5.3.1)$$

The tensor $T_{\mu\nu}$ is symmetric, and satisfies the conservation equation

$$\nabla_\nu T^\nu{}_\mu = 0 \ . \qquad (5.3.2)$$

It holds that

$$T_{00} = \frac{1}{2}(\dot\phi^2 + |D\phi|^2) \geq 0 \ ,$$

where $D\phi$ denotes the space-gradient of ϕ.

Exercise 5.3.1 Show that $T_{\mu\nu}$ satisfies the *dominant energy condition:* $T_{\mu\nu}X^\mu Y^\nu \geq 0$ for all causal future directed vectors X, Y.

Throughout the remainder of this section we will be working in Minkowskian coordinates, in which the Minkowski metric η equals diag$(-1, +1, \ldots, +1)$. Then (5.3.1) reads

$$\partial_\nu T^\nu{}_\mu = 0 \ . \qquad (5.3.3)$$

Given any hypersurface $\mathscr{S}$ in Minkowski space-time, the *total energy-momentum vector* of $\mathscr{S}$ is defined as

$$p_\mu(\mathscr{S}) = \int_{\mathscr{S}} T_\mu{}^\nu n_\nu d^3x \ , \qquad (5.3.4)$$

where n^μ is the field of normals to $\mathscr{S}$. If n^μ is timelike, one chooses n^μ to be future-directed. (Note the minus sign above, which is related to the current signature $(-, +, \ldots)$.

We emphasize that these integrals might change in an uncontrollable way when using coordinates other than the manifestly Minkowskian ones, because of the free tensor index on p_μ. So, even though $T_{\mu\nu}$ is a tensor field, $p_\mu(\mathscr{S})$ is *not* a vector field but rather a collection of numbers. (For one, a vector field is always attached to a point, but where would $p_\mu(\mathscr{S})$ be attached?)

Stokes' theorem provides the key to understand the properties of p_μ: Indeed, this theorem asserts that for any *bounded* set Ω with piecewise differentiable boundary we have

$$\int_{\partial\Omega} T_\mu{}^\nu n_\nu = \int_\Omega \partial_\nu T_\mu{}^\nu \ ,$$

where n_μ is the field of appropriately chosen normals to $\partial\Omega$. Since the left-hand side vanishes, we conclude that

$$\boxed{\int_{\partial\Omega} T_\mu{}^\nu n_\nu = 0} \ . \tag{5.3.5}$$

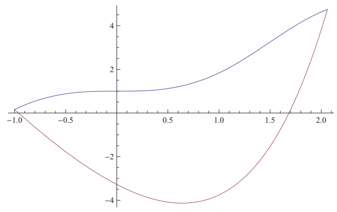

Figure 5.1: Two hypersurfaces in Minkowski space-time with timelike normal and common boundary.

As an application, in situations depicted in Figure 5.1 we obtain:

PROPOSITION 5.3.2 *Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be two hypersurfaces in Minkowski space-time with future-pointing timelike normals n^μ such that*

$$\mathscr{S}_1 \cup \mathscr{S}_2 = \partial\Omega \ . \tag{5.3.6}$$

for some bounded set Ω. Then

$$p_\mu(\mathscr{S}_1) = p_\mu(\mathscr{S}_2) \ , \tag{5.3.7}$$

PROOF: The result follows immediately from (5.3.5),

$$0 \quad = \quad \int_{\partial\Omega} T_\mu{}^\nu n_\nu = \int_{\mathscr{S}_2} T_\mu{}^\nu n_\nu - \int_{\mathscr{S}_1} T_\mu{}^\nu n_\nu = p_\mu(\mathscr{S}_1) - p_\mu(\mathscr{S}_2) \ ,$$

taking into account the orientation of the boundaries. □

It is often the case that when two bounded hypersurfaces $\mathscr{S}_1$ and $\mathscr{S}_2$, with compact closures, share a common boundary,

$$\partial\mathscr{S}_1 = \partial\mathscr{S}_2 \ ,$$

then one can find a bounded set Ω such that (5.3.6) holds.

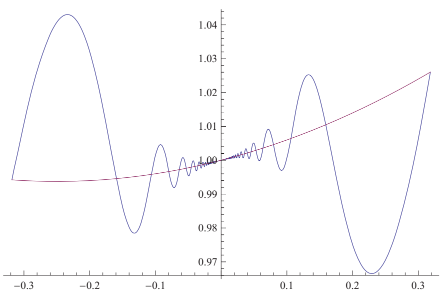

Figure 5.2: Two hypersurfaces in Minkowski space-time with common boundary which do not obviously bound a well behaved set. The normal direction is timelike in both cases, which is not immediately apparent because of different scaling of the axes.

In cases where the existence of such a set Ω is not completely clear, as e.g. in Figure 5.2, we proceed as follows: Note, first, that a hypersurface in Minkowski space-time with timelike normal is always a graph over a subset of $\{t = 0\} \approx \mathbb{R}^n$, hence there exist a bounded subset $\mathscr{U} \subset \mathbb{R}^n$ and functions $f_1, f_2 : \mathscr{U} \to \mathbb{R}$ such that

$$\mathscr{S}_a = \{t = f_a(\vec{x}) \ , \ \vec{x} \in \mathscr{U}\} \ .$$

(Actually, one needs to justify that the set $\mathscr{U}$ is the same for $\mathscr{S}_1$ and $\mathscr{S}_2$; this follows from the fact that both hypersurfaces are bounded and their boundaries coincide.) We let $M := \min(\inf_{\mathscr{U}} f_1, \inf_{\mathscr{U}} f_2) - 1$, and consider the sets

$$\Omega_a = \{\vec{x} \in \mathscr{U} \ , \ M \le t \le f_a(\vec{x})\}$$

Then

$$\partial\Omega_a = \mathscr{S}_a \cup \Sigma \ , \quad \text{where} \quad \Sigma = \{\vec{x} \in \partial\mathscr{U} \ , \ M \le t \le f_a(\vec{x})\} \cup \{\vec{x} \in \mathscr{U} \ , \ t = M\}$$

Applying (5.3.5) on Ω_a we obtain, with a convenient choice of normals,

$$p_\mu(\mathscr{S}_a) + p_\mu(\Sigma) = 0 \ , \ a = 1, 2 \ , \quad \text{whence} \quad p_\mu(\mathscr{S}_1) = p_\mu(\mathscr{S}_2) \ .$$

We conclude that:

Corollary 5.3.3 *For connected bounded hypersurfaces with boundary, $p_\mu(\mathscr{S})$ depends only upon $\partial\mathscr{S}$.*

One sometimes thinks of (5.3.7) as the statement that p_μ does not depend upon $\mathscr{S}$. However, care needs to be taken with such statements for hypersurfaces $\mathscr{S}$ extending to infinity. First, there are issues of convergence, to make sure that the integral defining p_μ is well defined and finite. Next, whether or not $p_\mu(\mathscr{S})$ depends upon $\mathscr{S}$ will depend upon the asymptotic behaviour of $\mathscr{S}$. The aim of what follows is to study this question in more detail for two specific families of hypersurfaces.

5.3.1 Conservation of energy on Minkowskian slices

We will consider only the simplest field configurations, where both $\phi|_{t=0}$ and $\partial_t\phi|_{t=0}$ are zero outside of a compact set, say $B(R)$. (Note that this does not apply directly to the linearized gravitational field, which decays relatively slowly at infinity. This is at the origin of many difficulties when trying to understand properly the issues arising, but we will not address those issues here.) In such a case

$$p_\mu(\{t=\tau\})$$

is well defined, and independent of τ. To see this, we start by noting the fundamental property of the wave equation that, for the solutions under consideration, for any τ the function $\phi|_{t=\tau}$ will be zero outside of $B(R+|\tau|)$. This is due to the fact that solutions of the wave equation propagate with the speed of light. Let then $\tau \in \mathbb{R}$ and choose $T \in \mathbb{R}^+$ such that $|\tau| \leq T$. We then have

$$p_\mu(\{t=\tau\}) \equiv \int_{\{t=\tau\}} T_\mu{}^\nu n_\mu d^3x = \int_{\{t=\tau,|\vec{x}|\leq R+T\}} T_\mu{}^\nu n_\mu d^3x \ .$$

Let

$$\Omega = \{\tau_1 \leq t \leq \tau_2 \ , \ |\vec{x}| \leq R+T\}$$

Using (5.3.5) we have

$$\begin{aligned}
0 &= \int_{\partial\Omega} T_\mu{}^\nu n_\nu \\
&= \int_{\{t=\tau_2,|\vec{x}|\leq R+T\}} T_\mu{}^\nu n_\nu - \int_{\{t=\tau_1,|\vec{x}|\leq R+T\}} T_\mu{}^\nu n_\nu + \underbrace{\int_{\{\tau_1\leq t\leq\tau_2 \ , \ |\vec{x}|=R+T\}} T_\mu{}^\nu n_\nu}_{0} \\
&= \int_{\{t=\tau_2\}} T_\mu{}^\nu n_\nu - \int_{\{t=\tau_1\}} T_\mu{}^\nu n_\nu \\
&= p_\mu(\{t=\tau_2\}) - p_\mu(\{t=\tau_1\}) \ .
\end{aligned}$$

Since T was arbitrary, we conclude that for all $\tau_1, \tau_2 \in \mathbb{R}$

$$p_\mu(\{t=\tau_2\}) = p_\mu(\{t=\tau_1\}) \ ;$$

equivalently,

$$\boxed{\tfrac{d}{d\tau} p_\mu(\{t=\tau\}) = 0} \ .$$

5.3.2 Energy-radiation on hypersurfaces asymptotic to light-cones

We pass next to the question of radiation of energy. Assuming again that both $\phi|_{t=0}$ and $\partial_t\phi|_{t=0}$ are zero outside of $B(R)$, one can show that the field ϕ satisfies the following property: there exists a function $f(u,\theta,\varphi)$ such that for $r\to\infty$ with $t-r$ being fixed it holds that

$$\phi(t,\vec x)=\frac{f(t-r,\theta,\varphi)}{r}+O(r^{-2})\ ,\quad \partial_\mu\phi(t,\vec x)=\partial_\mu\left(\frac{f(t-r,\theta,\varphi)}{r}\right)+O(r^{-3})\ . \tag{5.3.8}$$

We consider therefore hypersurfaces $\mathscr{S}_u$ such that $t-r$ approaches u along $\mathscr{S}_u$ as r goes to infinity. Such hypersurfaces are often called *hyperboloidal*, and they are asymptotic to the light-cones $t-r=u$. For such hypersurfaces we will see that the energy is not conserved, but is radiated away.

For simplicity we take the $\mathscr{S}_u$'s to be spherically symmetric graphs of the form

$$\mathscr{S}_u=\{t=\chi(r)+u\}\ ,\ \text{with}\ \lim_{r\to\infty}(\chi(r)-r)=0\ ; \tag{5.3.9}$$

the final result applies, however, to *any* family such that $t-r$ approaches u along $\mathscr{S}_u$ as r goes to infinity.

We choose $u_0\in\mathbb{R}$ and some large $R\geq 0$, and for $u\geq u_0$ we consider (5.3.5) on the set

$$\Omega=\{\chi(r)+u_0\leq t\leq\chi(r)+u\ ,\ r\leq R\}\ .$$

Hence

$$\begin{aligned}0 &= \int_{\partial\Omega} T_\mu{}^\nu n_\nu\\ &= \int_{\mathscr{S}_u\cap\{r\leq R\}} T_\mu{}^\nu n_\nu-\int_{\mathscr{S}_{u_0}\cap\{r\leq R\}} T_\mu{}^\nu n_\nu\\ &\quad +\int_{\{\chi(R)+u_0\leq t\leq\chi(R)+u\ ,\ |\vec x|=R\}} T_\mu{}^\nu n_\nu\ .\end{aligned} \tag{5.3.10}$$

The last integral above equals

$$\int_{t=\chi(R)+u_0}^{\chi(R)+u}\int_{S(R)} T_\mu{}^i n_i d^2S dt\ ,$$

and so

$$\frac{d}{du}\int_{\mathscr{S}_u\cap\{r\leq R\}} T_\mu{}^\nu n_\nu=-\int_{S(R)} T_\mu{}^i n_i d^2S\ . \tag{5.3.11}$$

Passing to the limit $R\to\infty$ we obtain (happily assuming that the derivative of the limit is the limit of the derivative...)

$$\frac{d}{du}p_\mu(\mathscr{S}_u)=-\lim_{R\to\infty}\int_{S(R)} T_\mu{}^i n_i d^2S\ . \tag{5.3.12}$$

To calculate the right-hand side with $\mu=0$, we use the fact that the relevant normal direction is $n_i=n^i=-\frac{x^i}{r}$; •5.3.1 the sign is determined by a more careful •5.3.1: **ptc:** *the sign is correct now* analysis of the problem at hand, see Remarks 5.3.4 and 5.3.5 below. Thus

$$T_0{}^i n_i = -\partial_t\phi\partial_i\phi\frac{x^i}{r} = -\partial_t\phi\partial_r\phi \;.$$

Using (5.3.8) we obtain

$$\begin{aligned} T_0{}^i n_i &= -\partial_t\phi\partial_r\phi \\ &= -\partial_t\left(\frac{f(t-r,\theta,\varphi)}{r} + O(r^{-2}))\right)\partial_r\left(\frac{f(t-r,\theta,\varphi)}{r} + O(r^{-2}))\right) \\ &= \frac{\dot f^2}{r^2} + O(r^{-3}) \;, \text{ where } \dot f = \partial_u f \;. \end{aligned}$$

Hence

$$\frac{d}{du}p_0(\mathscr{S}_u) = -\int_{S(1)} \dot f^2 d^2S \;. \tag{5.3.13}$$

We see that the energy decreases, with energy flux equal to $-\dot f^2$.

Remark 5.3.4 For those with some knowledge of differential forms, we recall that in (5.3.5) one integrates $T_\mu{}^\nu dS_\nu$ over $\partial\Omega$, where the three-forms dS_ν are defined as

$$dS_\nu = \sqrt{|\det g_{\alpha\beta}|}\partial_\nu \rfloor dx^0 \wedge \cdots \wedge dx^n \;.$$

For the last integral in (5.3.10) it is convenient to go to spherical coordinates, since $dr = 0$ on the set integrated upon. Then the only form above which gives a non-trivial contribution to the integral is

$$dS_r = r^2 \sin\theta\partial_r \rfloor dt\wedge dr\wedge d\theta\wedge d\varphi = -r^2\sin\theta\, dt\wedge d\theta\wedge d\varphi \;.$$

The last minus sign explains the minus sign in the formula $n^i = -x^i/r$. □

Remark 5.3.5 For those who have never heard of exterior algebra, it appears that the only way to explain the minus sign in (5.3.12) is to rederive this equation from scratch, using only vector calculus in an Euclidean $\mathbb{R}^3$. We will, however, still need to know that for hypersurfaces that are graphs over a subset of $\mathbb{R}^3$ the equality $p_\mu(\mathscr{S}_1) = p_\mu(\mathscr{S}_2)$ holds, compare (5.3.7). This can be established by calculations similar to the ones that we are about to carry-out, and is left as an exercice.

Given (5.3.7), we start by noting that for the hypersurfaces (5.3.9) we have

$$p_\mu(\mathscr{S}_u \cap (\{|\vec x| \le R\}) = p_\mu(\{t = \chi(R) + u, |\vec x| \le R\}) \;, \tag{5.3.14}$$

as both those hypersurfaces are graphs over $B(R)$ and share the common boundary $\{t = \chi(R) + u, |\vec x| = R\}$. We then consider the "space-time cylinder"

$$\Omega = \{\chi(R) + u_0 \le t \le \chi(R) + u, |\vec x| \le R\} \;,$$

and the identity

$$0 = \int_\Omega \partial_\nu T_\mu{}^\nu d^4x = \int_\Omega (\partial_0 T_\mu{}^0 + \partial_i T_\mu{}^i) d^4x \; . \qquad (5.3.15)$$

The $\partial_0 T_\mu{}^0$ terms integrate to

$$\begin{aligned}
\int_\Omega \partial_0 T_\mu{}^0 d^4x &= \int_{t=\chi(R)+u_0}^{\chi(R)+u} \int_{B(R)} \partial_0 T_\mu{}^0 d^3x \, dt \\
&= \int_{B(R)} T_\mu{}^0(t=\chi(R)+u, \vec{x}) d^3x - \int_{B(R)} T_\mu{}^0(t=\chi(R)+u_0, \vec{x}) d^3x \\
&= -\int_{B(R)} T_{\mu 0}(t=\chi(R)+u, \vec{x}) d^3x + \int_{B(R)} T_{\mu 0}(t=\chi(R)+u_0, \vec{x}) d^3x \\
&= -p_\mu(\{t=\chi(R)+u, |\vec{x}| \le R\}) + p_\mu(\{t=\chi(R)+u_0, |\vec{x}| \le R\}) \\
&= -p_\mu(\mathscr{S}_u \cap (\{|\vec{x}| \le R\}) + p_\mu(\mathscr{S}_{u_0} \cap (\{|\vec{x}| \le R\}) \; ;
\end{aligned}$$

in the before last-step we have used $n^\mu \partial_\mu = \partial_0$, hence $T_{\mu\nu} n^\nu = T_{\mu 0}$, while in the last step (5.3.14) was invoked. Note the change of signs arising from the fact that $T_{\mu 0} = -T_\mu{}^0$.

Using the divergence theorem on $B(R)$, the $\partial_i T_\mu{}^i$ term in (5.3.15) integrates to

$$\begin{aligned}
\int_\Omega \partial_i T_\mu{}^i d^4x &= \int_{t=\chi(R)+u_0}^{\chi(R)+u} \int_{B(R)} \partial_i T_\mu{}^i d^3x \, dt \\
&= \int_{t=\chi(R)+u_0}^{\chi(R)+u} \int_{S(R)} T_\mu{}^i \frac{x^i}{r} d^2S \; .
\end{aligned}$$

Collecting all this, we obtain

$$p_\mu(\mathscr{S}_u \cap (\{|\vec{x}| \le R\}) = p_\mu(\mathscr{S}_{u_0} \cap (\{|\vec{x}| \le R\}) + \int_{t=\chi(R)+u_0}^{\chi(R)+u} \int_{S(R)} T_\mu{}^i \frac{x^i}{r} d^2S \; ,$$

This gives (5.3.11), with n^i equal to $-x^i/r$, as claimed. □

5.4 The quadrupole formula

Now, recall that the linearized gravitational field $\bar{h}_{\mu\nu}$ satisfies

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \; .$$

This leads to the retarded solutions •5.4.1

$$\bar{h}_{\mu\nu}(t, \vec{x}) = \frac{4G}{c^4} \int_{\mathbb{R}^3} \frac{T_{\mu\nu}(t - |\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|} d^3y \; .$$

•5.4.1: **ptc:** *please note that there was an overall sign wrong in this formula in the lecture where the quadrupole formula for $\bar{h}_{ij}$ was derived, corrected*

In any case, outside of the sources we have

$$\Box\bar{h}_{\mu\nu} = 0 \ .$$

So, one expects that a formula in the spirit of (5.3.12) should apply in this case.

For an isolated system we have derived, at large distances,

$$\bar{h}_{00} \approx \frac{4GM}{c^4 r} \ , \quad \bar{h}_{0i} \approx 0 \ , \quad \bar{h}_{ij} \approx \frac{2G}{3c^2 r}\ddot{q}_{ij}(t-r)$$

where M is the total mass and q_{ij} the quadrupole moments of the energy density:

$$M(\tau) = \int_{t=\tau} T_{00} d^3x = M(0) =: M \ , \quad q_{ij}(\tau) = 3\int_{t=\tau} T_{00} x^i x^j d^3x \ .$$

So, one expects that the energy flux will be proportional to some combination of the squares of third derivatives of q_{ij}. A more careful analysis leads to the *Einstein quadrupole formula*, first derived by Einstein in 1917,

$$\frac{d}{du} p_0(\mathscr{S}_u) = -\frac{G}{45c^5}\sum_{ij}\left(\frac{d^3 Q_{ij}}{du^3}\right)^2 \ , \quad \text{where } Q_{ij} = q_{ij} - \frac{1}{3}q^k{}_k\delta_{ij} \ . \tag{5.4.1}$$

5.5 Multipole expansions

We return now to the weak-field formula

$$\bar{h}_{\mu\nu}(t,\vec{x}) = \kappa\int_{\mathbb{R}^3} \frac{T_{\mu\nu}(t-|\vec{x}-\vec{y}|,\vec{y})}{|\vec{x}-\vec{y}|} d^3y \ . \tag{5.5.1}$$

We continue to assume that we are located at a very large distance from the sources. So far we made the approximation that

$$|\vec{x}-\vec{y}| \approx |\vec{x}| \tag{5.5.2}$$

leading to

$$\bar{h}_{0\mu}(t,\vec{x}) \approx \frac{\kappa}{|\vec{x}|}\int_{\mathbb{R}^3} T_{00}(t-|\vec{x}|,\vec{y})d^3y = \frac{\kappa}{|\vec{x}|}\int_{\mathbb{R}^3} T_{0\mu}(0,\vec{y})d^3y = \frac{\kappa p_\mu}{|\vec{x}|} \ ,$$

where p_μ is the total four-momentum of matter fields; the second equality above is due to the Minkowskian conservation equation $\partial_\mu T^\mu{}_\nu = 0$.

We can make a boost of the coordinates to bring p_μ to the form $(m,\vec{0})$; this defines the *zero-momentum frame*. In this frame we have

$$\bar{h}_{00}(t,\vec{x}) \approx = \frac{\kappa M}{|\vec{x}|} \ , \quad \bar{h}_{0i}(t,\vec{x}) \approx 0 \ . \tag{5.5.3}$$