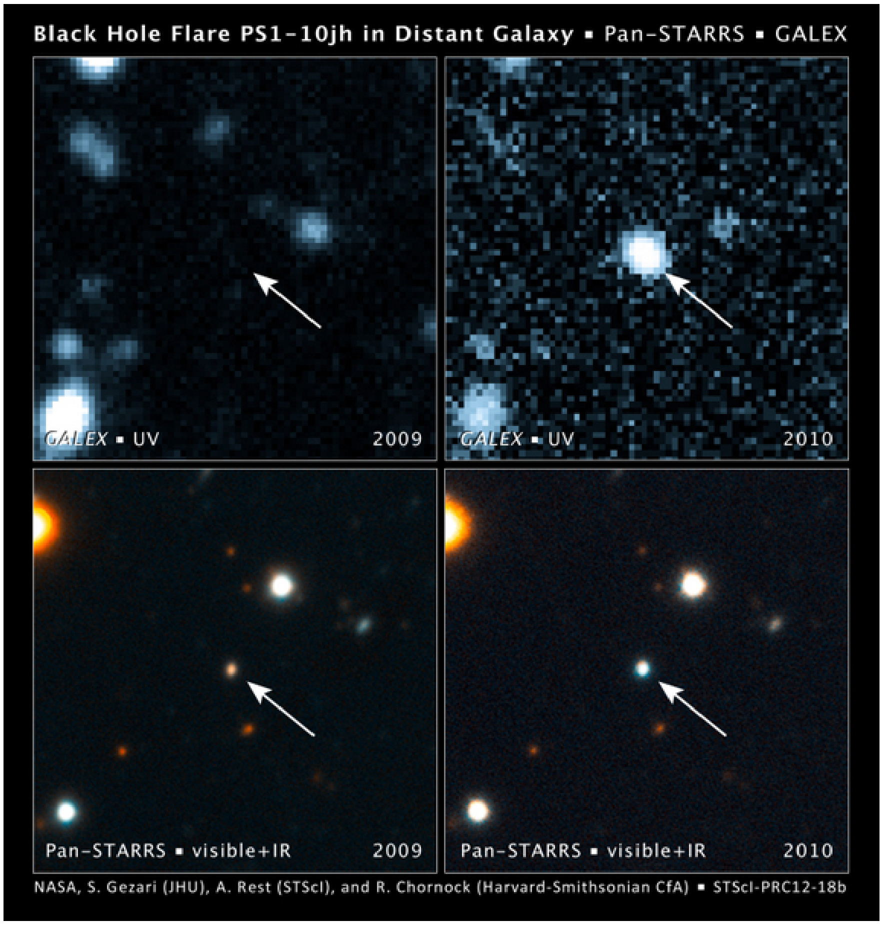

May 3, 2012:

Some news: luminous ultraviolet-optical flare from the nuclear region of an inactive galaxy at a redshift of 0.1696 on March 28, 2011, in the constellation Draco, 2?-4? million light years away, whose central black hole is 4 million times as massive as the sun. The galaxy was 350 times brighter in UV in June 2011 as before the flare.

Such an event might happen once every 10,000-100,000 years in a galaxy with a supermassive black hole in the center

“An ultraviolet-optical flare from the tidal disruption of a helium-rich stellar core”, Suvi Gezari et al.,
`doi:10.1038/nature10990`

“Black holes: Star ripped to shreds, Giuseppe Lodato, `doi:10.1038/nature11191`

Geodetic precession, Reminders:

Gyroscope equation:

$$\frac{Ds}{d\tau} = 0 \ , \quad g(u,s) = 0 \ , \tag{1}$$

$$\frac{Ds^\mu}{d\tau} := \frac{ds^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} s^\alpha \dot\gamma^\beta \ .$$

$$\boxed{\frac{d}{d\tau}(g(X,Y)) = g(\frac{DX}{d\tau}, Y) + g(X, \frac{DY}{d\tau}) \ .}$$

§II.10.2: Geodetic precession on circular geodesics in Schwarzschild

Gravity Probe B: C.W.F Everitt et al, arXiv:1105.3456 [gr-qc] (see also C.M. Will, arXiv:1106.1198 [gr-qc])

We consider the gyroscope equations on affinely parameterized timelike circular geodesics in Schwarzschild on the $\{z = 0\}$ plane:

$$\gamma(\tau) = (t(\tau), R, \frac{\pi}{2}, \varphi(\tau)) \qquad \Longrightarrow \qquad u^r = 0 = u^\theta \ .$$

We have

$$0 = g(s,u) = -\left(1 - \frac{2m}{r}\right) s^t u^t + r^2 s^\varphi u^\varphi \ ,$$

and since $u^t \neq 0$ for timelike geodesics, we can calculate s^t if s^φ is known:

$$s^t = \frac{r^2 u^\varphi}{\left(1 - \frac{2m}{r}\right) u^t} s^\varphi \ . \tag{2}$$

Explicitly, the parallel transport equation for s reads

$$\begin{aligned} \frac{ds^\alpha}{d\tau} &= -\Gamma^\alpha{}_{\mu\nu} s^\mu u^\nu \\ &= -\Gamma^\alpha{}_{\mu t} s^\mu u^t - \Gamma^\alpha{}_{\mu\varphi} s^\mu u^\varphi \ . \end{aligned} \tag{3}$$

We need to calculate the required Christoffels, which have

1. either at least one φ lower index, or
2. at least one t lower index.

For comparison purposes, note that the parallel transport calculations below

apply for Minkowski by setting $m = 0$

for a circular orbit (NOT GEODESIC!) so (2) needs revisiting: In Minkowski we use

$$\begin{aligned} \Gamma^t_{\mu\nu} &= \frac{1}{2} g^{t\alpha}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}) \\ &= \frac{1}{2} g^{tt}(\partial_\mu g_{\nu t} + \partial_\nu g_{\mu t} - \partial_t g_{\mu\nu}) \\ &= 0 \end{aligned}$$

hence we obtain instead

$$\frac{ds^t}{d\tau} = 0 \text{ when } m = 0 \ .$$

In Schwarzschild we start with the s^θ equation:

$$\begin{aligned} \Gamma^\theta{}_{\mu\nu} &= \frac{1}{2} g^{\theta\alpha} \left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right) \\ &= \frac{1}{2} g^{\theta\theta} \left(\partial_\mu g_{\nu\theta} + \partial_\nu g_{\mu\theta} - \partial_\theta g_{\mu\nu}\right) \end{aligned}$$

and since

$$g_{t\theta} = 0 = g_{t\varphi} = \partial_\theta g_{tt}$$

the only relevant not-obviously-vanishing component is

$$\partial_\theta g_{\varphi\varphi} = \partial_\theta (r^2 \sin^2\theta) = 2r^2 \sin\theta \cos\theta \ ,$$

which also vanishes at $\theta = \pi/2$. Hence

$$\frac{ds^\theta}{d\tau} = 0 \ ,$$

and s^θ is constant.

We continue with s^r:

$$\begin{aligned}\Gamma^r{}_{\mu t} &= \frac{1}{2}g^{r\alpha}\big(\partial_\mu g_{t\alpha} + \underbrace{\partial_t g_{\mu\alpha}}_{0} - \partial_\alpha g_{\mu t}\big) \\ &= \frac{1}{2}g^{rr}\big(\underbrace{\partial_\mu g_{tr}}_{0} - \partial_r g_{\mu t}\big) = -\frac{1}{2}g^{rr}\partial_r g_{tt}\delta^t_\mu \ .\end{aligned}$$

$$\begin{aligned}\Gamma^r{}_{\mu\varphi} &= \frac{1}{2}g^{rr}\big(\partial_\mu \underbrace{g_{\varphi r}}_{0} + \underbrace{\partial_\varphi g_{\mu r}}_{0} - \partial_r g_{\mu\varphi}\big) \\ &= -\frac{1}{2}g^{rr}\partial_r g_{\varphi\varphi}\delta^\varphi_\mu \ .\end{aligned}$$

Thus

$$\begin{aligned}\frac{ds^r}{d\tau} &= -\Gamma^r{}_{\mu t}s^\mu u^t - \Gamma^r{}_{\mu\varphi}s^\mu u^\varphi \\ &= -\Gamma^r{}_{tt}s^t u^t - \Gamma^r{}_{\varphi\varphi}s^\varphi u^\varphi \\ &= \frac{1}{2}g^{rr}\left(\partial_r g_{tt}s^t u^t + \partial_r g_{\varphi\varphi}s^\varphi u^\varphi\right) \ .\end{aligned}$$

In Schwarzschild we use the $s^t - s^\varphi$ relation (2),

$$s^t = \frac{r^2 u^\varphi}{\left(1-\frac{2m}{r}\right)u^t}s^\varphi \ . \tag{2}$$

and

$$g^{rr} = 1-\frac{2m}{r} \ , \quad g_{tt} = -1+\frac{2m}{r} \ , \quad \partial_r g_{tt} = -\frac{2m}{r^2} \ , \quad \partial_r \underbrace{g_{\varphi\varphi}}_{r^2} = 2r \ ,$$

to obtain

$$\frac{ds^r}{d\tau} = \frac{1}{2}\left(1-\frac{2m}{r}\right)\left(-\frac{2m}{r^2}\frac{r^2}{\left(1-\frac{2m}{r}\right)} + 2r\right)s^\varphi u^\varphi$$

$$
\begin{aligned}
&= \left(-\frac{m}{r^2}r^2 + r\left(1-\frac{2m}{r}\right)\right) s^\varphi u^\varphi \\
&= (r-3m)s^\varphi u^\varphi \ .
\end{aligned}
$$

$$
\boxed{\frac{ds^r}{d\tau} = (r-3m)s^\varphi u^\varphi \ .} \tag{4}
$$

In Minkowski we use

$$
\Gamma^r_{tt} = -\frac{1}{2}g^{r\alpha}(\partial_t g_{tr} + \partial_t g_{tr} - \partial_r g_{tt}) = 0
$$

to conclude that the above remains correct when $m = 0$.

$$
\frac{ds^r}{d\tau} \ = \ (r-3m)s^\varphi u^\varphi \ ,
$$

Since

$$
u^\varphi = \frac{d\varphi}{d\tau} = \frac{dt}{d\tau}\frac{d\varphi}{dt} = u^t\Omega \ ,
$$

where, in the Schwarzschild case,

$$
\Omega = \sqrt{\frac{m}{r^3}}
$$

is the coordinate angular velocity of the orbit (exercice!), we obtain the desired equation for s^r,

$$
\begin{aligned}
\frac{ds^r}{dt} &= \frac{d\tau}{dt}\frac{ds^r}{d\tau} = \frac{1}{u^t}\frac{ds^r}{d\tau} = (r-3m)s^\varphi\frac{u^\varphi}{u^t} \\
&= (r-3m)\Omega s^\varphi \ .
\end{aligned}
$$

$$
\boxed{\frac{ds^r}{dt} = (r-3m)\Omega s^\varphi \ , \quad \frac{u^\varphi}{u^t} = \Omega \ .} \tag{5}
$$

We continue with the s^φ equation. For this we calculate

$$\begin{aligned}
\Gamma^\varphi{}_{\mu t} &= \frac{1}{2} g^{\varphi\alpha}\big(\partial_\mu g_{t\alpha} + \underbrace{\partial_t g_{\mu\alpha}}_{0} - \partial_\alpha g_{\mu t}\big) \\
&= \frac{1}{2} g^{\varphi\varphi}\big(\partial_\mu g_{t\varphi} - \partial_\varphi g_{\mu t}\big) = 0 \;.
\end{aligned}$$

$$\begin{aligned}
\Gamma^\varphi{}_{\mu\varphi} &= \frac{1}{2} g^{\varphi\varphi}\big(\partial_\mu g_{\varphi\varphi} + \partial_\varphi g_{\mu\varphi} - \partial_\varphi g_{\mu\varphi}\big) \\
&= \frac{1}{2} g^{\varphi\varphi}\partial_\mu g_{\varphi\varphi} \;.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{ds^\varphi}{d\tau} &= -\Gamma^\varphi{}_{\mu t} s^\mu u^t - \Gamma^\varphi{}_{\mu\varphi} s^\mu u^\varphi = -\frac{1}{2} g^{\varphi\varphi}\partial_r g_{\varphi\varphi} s^r u^\varphi \\
&= -\frac{1}{r} s^r u^\varphi \;. \qquad (6)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{ds^\varphi}{dt} &= \underbrace{\frac{d\tau}{dt}}_{\frac{1}{u^t}} \frac{ds^\varphi}{d\tau} = -\frac{1}{r} s^r \frac{u^\varphi}{u^t} \\
&= -\frac{\Omega}{r} s^r \;.
\end{aligned}$$

$$\boxed{\frac{ds^\varphi}{dt} = -\frac{\Omega}{r} s^r \;.} \qquad (7)$$

Summarising,

$$\frac{ds^r}{dt} = (r-3m)\Omega s^\varphi \ , \quad \frac{ds^\varphi}{dt} = -\frac{\Omega}{r} s^r \ . \tag{8}$$

and in Minkowski the same equation with $m = 0$:

$$\frac{ds^r}{dt} = r\Omega s^\varphi \ , \quad \frac{ds^\varphi}{dt} = -\frac{\Omega}{r} s^r \ . \tag{9}$$

Differentiating the first equation and using the second one we find

$$\frac{d^2 s^r}{dt^2} = -\frac{(r-3m)\Omega^2}{r} s^r \ , \tag{10}$$

with an identical equation for s^φ. This is a harmonic oscillator with frequency

$$\omega = \sqrt{1-\frac{3m}{r}}\Omega \approx \left(1-\frac{3m}{2r}\right)\Omega \ , \tag{11}$$

where the second, approximate, equality holds for small m/r.

Could we have guessed the result in Minkowski?

frequency *smaller* than the frequency of the orbit

This effect is known under the name of *geodetic precession*, first predicted by Willem de Sitter in 1916, in the Earth-Moon system's motion.

§II.11: Fermi-Walker transport, Thomas precession

Given a proper-time-parameterised world-line $\gamma^\mu(\tau)$, with unit tangent u and acceleration four-vector $a := Du/d\tau \neq 0$, we would like to describe the motion of a gyroscope subject to no external forces other than those that keep it on the world-line *without torque*. Keeping in mind that a gyroscope is described by a vector s satisfying $g(s,u) = 0$, the parallel transport equation is not adequate since then

$$\frac{d(g(u,s))}{d\tau} = g(\frac{Du}{d\tau}, s) + g(u, \frac{Ds}{d\tau}) = g(a,s) \ ,$$

which is not zero in general. The simplest modification to the parallel transport equation is to write

$$\frac{Ds}{d\tau} = \phi u \ .$$

for some function ϕ. One then has

$$\begin{aligned} \frac{d(g(u,s))}{d\tau} &= g(a,s) + g(u, \underbrace{\frac{Ds}{d\tau}}_{\phi u}) \\ &= g(a,s) + \phi \underbrace{g(u,u)}_{=-1} \ . \end{aligned}$$

This will vanish if we choose $\phi = g(a,s)$. So a possible equation reads

$$\frac{Ds}{d\tau} = g(s,a)u \ . \tag{12}$$

The equation leads to an effect known as the *Thomas precession* even in Minkowski space-time.

There appears to be some confusion about the exact meaning of "Thomas precession". Some authors tie it to the fact, that the composition of two boost transformations with velocities u and v is a Lorentz transformation which is the composition of a boost and a rotation. The rotation part of this composition is then called Thomas precession.

Equation (12) is a special case of the *Fermi-Walker transport* of any vector s along a world-line γ, not necessarily orthogonal to the four-velocity. The Fermi-Walker transport equation reads

$$\frac{Ds^\mu}{d\tau} = g(s,a)u^\mu - g(s,u)a^\mu \ . \tag{13}$$

This equation has interesting features. First,

$$\frac{Du^\mu}{d\tau} = \underbrace{g(u,a)}_{=0}u^\mu - g(u,u)a^\mu = a^\mu \ , \tag{14}$$

which shows that the four-velocity is FW-transported along γ. Next, consider two vectors, say X and Y, which are FW-transported along γ, then

$$\begin{aligned}
\frac{d(g(X,Y))}{d\tau} &= g(\frac{DX}{d\tau},Y) + g(X,\frac{DY}{d\tau}) \\
&= g(g(X,a)u - g(X,u)a, Y) + g(X, g(Y,a)u - g(Y,u)a) \\
&= g(X,a)g(u,Y) - g(X,u)g(a,Y) \\
&\quad + g(Y,a)g(X,u) - g(Y,u)g(X,a) \\
&= 0 \ .
\end{aligned}$$

Hence, Fermi-Walker transport preserves scalar products. In particular a vector initially orthogonal to u will remain orthogonal when FW transported.

Finally, Fermi-Walker transport reduces to parallel transport for geodesics.

It is argued in the literature that the Fermi-Walker equation provides the right equation for describing the motion of gyroscopes along general worldlines.

While all the facts above are related, it is far from clear that they are equivalent in any sense. For instance, a gyroscope is a complicated object, and its behaviour under relativistic accelerations and velocities is by no means obvious.

§II.12: The Lense-Thirring effect

When studying linearized gravity we have derived the following leading-order behaviour of the metric for weak gravitational fields:

$$g_{00} \approx -1 + \frac{2m}{r} , \qquad g_{ij} \approx (1 + \frac{2m}{r})\delta_{ij} .$$

This is consistent with the Schwarzschild metric in *isotropic coordinates* (cf. PS4)

$$g \;=\; -(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}})^2 dt^2 + (1 + \frac{m}{2r})^4 (dx^2 + dy^2 + dz^2)$$

So, the Schwarzschild calculations give a good approximation for the motion of gyroscopes in all weak gravitational fields at large distances.

It turns out that a *rotating source* will give a contribution to the gravitational field of the form

$$g_{0i} \approx -2\epsilon_{ijk} \frac{x^i J^k}{r^3} ,$$

where J^k is the angular-momentum vector of the source, and we will prove this in a few lectures.

The inclusion of such terms in the gyroscope equation gives rise to a new effect, first discussed by Lense and Thirring, known as *frame dragging*, or as the Lense-Thirring effect. •0.0.1

•0.0.1: **ptc:** *I need to clarify this*

Toy model

$$g = -c^2 dt^2 + dx^2 + dy^2 + dz^2 - \frac{4GJ}{c^3 r^2}(c\,dt)(\frac{xdy - ydx}{r}) . \tag{15}$$

along the curve

$$\gamma(\tau) = (x^\mu(\tau)) = (t(\tau), 0, 0, z(\tau)) \ .$$

In order to see that this is a geodesic,

$$\frac{d^2 x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \ ,$$

we need those Christoffels where one of the α and β equals t or z.

Now along γ we have $g_{\mu\nu} = \eta_{\mu\nu}$ and further

$$\Gamma^\mu_{z\alpha}|_\gamma \ = \ \frac{1}{2}\eta^{\mu\nu}(\partial_z g_{\nu\alpha} + \partial_\alpha \underbrace{g_{\nu z}}_{0 \text{ or } 1} - \partial_\nu g_{z\alpha}) = 0 \ .$$

Indeed,

$$\partial_z g_{ty}|_\gamma = \partial_z \left(-\frac{4GJx}{c^2 r^3} \right) |_\gamma = x \partial_z \left(-\frac{4GJ}{c^2 r^3} \right) |_\gamma = 0 \ ,$$

similarly for $\partial_z g_{tx}|_\gamma$, while the remaining ∂_z derivatives of $g_{\alpha\nu}$ obviously vanish. Further,

$$\begin{aligned}
\Gamma^\mu_{0\alpha}|_\gamma \ &= \ \frac{1}{2}\eta^{\mu\nu}(\partial_0 g_{\nu\alpha} + \partial_\alpha g_{\nu 0} - \partial_\nu g_{0\alpha}) \\
&= \ \frac{1}{2}\eta^{\mu\nu}(\partial_\alpha g_{\nu 0} - \partial_\nu g_{0\alpha}) \ , \qquad (16) \\
\Gamma^\mu_{00}|_\gamma \ &= \ \frac{1}{2}\eta^{\mu\nu}(\partial_0 g_{\nu 0} - \partial_\nu g_{00}) = 0 \ ,
\end{aligned}$$

and so this is indeed an affinely parameterized geodesic when

$$(x^\mu(\tau)) = (u^t, 0, 0, u^z)\tau \ ,$$

with u^t and u^z being constants satisfying $u^t = \sqrt{1 + (u^z)^2}$.

Along γ the orthogonality relation $g(s,u) = 0$ simply reads

$$u^t s^t = u^z s^z \qquad \Longleftrightarrow \qquad s^t = \frac{u^z}{u^t} s^z \ . \tag{17}$$

The z-component of the parallel-transport equation reads

$$0 = \frac{ds^z}{d\tau} + \Gamma^z_{\alpha\beta} u^\alpha s^\beta = \frac{ds^z}{d\tau} + \Gamma^z_{0\beta} u^t s^\beta = \frac{ds^z}{d\tau} \ ,$$

and s^z is constant. Note that if $s^z(0) = 0$, then both s^z and s^t vanish for all τ.

We choose for simplicity $s = (0, s^x, s^y, 0)$. One then has

$$0 = \frac{ds^x}{d\tau} + \Gamma^x_{\alpha\beta} u^\alpha s^\beta = \frac{ds^x}{d\tau} + \Gamma^x_{0x} u^t s^x + \Gamma^x_{0y} u^t s^y \tag{18}$$

Recall (16)

$$\Gamma^\mu_{0\alpha}|_\gamma \ = \ \frac{1}{2}\eta^{\mu\nu}(\partial_\alpha g_{\nu 0} - \partial_\nu g_{0\alpha}) \ , \tag{16}$$

so

$$\begin{aligned}
\Gamma^x_{0x}|_\gamma \ &= \ \frac{1}{2}\eta^{xx}(\partial_x g_{x0} - \partial_x g_{0x}) = 0 = \Gamma^y_{0y}|_\gamma \ , \\
\Gamma^x_{0y}|_\gamma \ &= \ \frac{1}{2}\eta^{x}(\partial_y g_{x0} - \partial_x g_{0y}) \ ,
\end{aligned}$$

$$\partial_x g_{0y}|_\gamma = \partial_x \left(-\frac{4GJx}{c^2 r^3} \right) |_\gamma = -\frac{4GJ}{c^2 r^3}|_\gamma = -\partial_x g_{0y}|_\gamma \ ,$$

$$\Gamma^x_{0y}|_\gamma = \frac{4GJ}{c^2 r^3}|_\gamma \ ,$$

Coming back to (18)

$$\begin{aligned}
0 \ &= \ \frac{ds^x}{d\tau} + \Gamma^x_{0x} u^t s^x + \Gamma^x_{0y} u^t s^y \\
&= \ \frac{ds^x}{d\tau} + \frac{4GJ}{c^2 r^3} u^t s^y \ .
\end{aligned}$$

A similar calculation, or by symmetry,

$$0 \ = \ \frac{ds^y}{d\tau} - \frac{4GJ}{c^2 r^3} u^t s^x \ .$$

For large distances and small radial velocities, so that $r \approx$ const, $u^t \approx 1$, we view

$$\omega := \frac{4GJ}{c^2 r^3}$$

as a constant, and so

$$\frac{ds^x}{d\tau} = -\omega s^y \ , \qquad \frac{ds^y}{d\tau} = \omega s^x \ ,$$

hence

$$\frac{d^2 s^x}{d\tau^2} = -\omega^2 s^x \ , \qquad \frac{d^2 s^y}{d\tau^2} = -\omega^2 s^y \ ,$$

and (s^x, s^y) is precessing with angular velocity ω .

The experimental verification of the frame-dragging formula has been carried out by the Gravity Probe B. The formula was confirmed up to a statistical uncertainty of 14% and a systematic uncertainty of 10%. See `http://einstein.stanford.edu/highlights/status1.html#frame-dragging` for more information.