April 26, 2012
Geodesics in Schwarzschild, Reminders:

$$\left(\frac{dr}{ds}\right)^2 = -\left(\lambda + \frac{J^2}{r^2}\right)(1 - \frac{2m}{r}) + E^2 \,. \qquad (1)$$

where $\lambda = 0$ for null geodesics, and $\lambda = 1$ for timelike ones, and

$$\frac{dt}{ds} = \frac{E}{1 - \frac{2m}{r}} \,. \qquad (2)$$

$$\frac{d\varphi}{ds} = \frac{J}{r^2} \,. \qquad (3)$$

$u := m/r$, if $J \neq 0$:

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{m^2E^2}{J^2} - u^2(1 - 2u) - \frac{\lambda m^2(1 - 2u)}{J^2} \,, \qquad (4)$$

$$\frac{d^2u}{d\varphi^2} = -u + 3u^2 + \frac{\lambda m^2}{J^2} \,. \qquad (5)$$

§II.9 Massive test particles: advance of the perihelion/periastron

Periastron: the point at which the orbit is closest to the star.

When the star is our sun: *perihelion*.

Aim of this section: compare the motion in the Schwarzschild metric with the Keplerian orbits, at distances large compared to m.

The orbit of the earth ($r = r_{♁}$, $J = J_{♁}$) around the sun (so that $m = M_{☼}$) we have

$$\frac{2M_{☼}}{r_{♁}} = 2u \sim 10^{-8} \ , \quad \frac{J_{♁}^2}{r_{♁}^2} = J_{♁}^2 M_{☼}^{-2} u^2 \sim 10^{-8} \ , \quad \frac{M_{☼}}{J_{♁}^2} \sim 10^{-8} \ .$$

(m is the mass of the central body, J is the angular momentum per unit mass of the orbiting body)

Exercice: check the following numbers, work out the corresponding numbers for Mars and Venus

Comparison with Newton, once again:

Return to the Schwarzschildian (1) with $\lambda = 1$

$$\begin{aligned} \left(\frac{dr}{ds}\right)^2 &= -\left(1 + \frac{J^2}{r^2}\right)(1 - \frac{2m}{r}) + E^2 \\ &= \underbrace{E^2 - 1 - \frac{J^2}{r^2} + \frac{2m}{r}}_{\text{leading contribution}} + \underbrace{\frac{2mJ^2}{r^3}}_{\frac{m}{r}\times\frac{J^2}{r^2}} \ . \end{aligned} \tag{6}$$

Newtonian test body with unit mass, and with angular momentum J_N, conservation of angular-momentum leads to the following formula for the Newtonian energy E_N

$$E_N = \frac{1}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - \frac{m}{r} = \frac{1}{2}(\dot{r}^2 + \frac{J_N^2}{r^2}) - \frac{m}{r} \ ,$$

hence

$$\left(\frac{dr}{dt}\right)^2 = 2E_N - \frac{J_N^2}{r^2} + \frac{2m}{r} \ . \tag{7}$$

Identical equations

1. if $s = t$ and if
2. we neglect the $\frac{2mJ^2}{r^3}$ term in (6),
3. identify $E^2 - 1$ with $2E_N$
4. identify J with J_N (recall unit mass)

Similarly the u equation (4) with $\lambda = 1$,

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{m^2E^2}{J^2} - u^2(1-2u) - \frac{m^2(1-2u)}{J^2} , \qquad (8)$$

should be compared with its version where the u^3 terms have been dropped out:

$$\begin{aligned} \left(\frac{du}{d\varphi}\right)^2 &= \frac{m^2E^2}{J^2} - u^2 - \frac{m^2(1-2u)}{J^2} \\ &= \frac{m^2}{J^2}(E^2 - 1 + 2u) - u^2 \end{aligned}$$

Recall the Newtonian equations from a previous lecture, where now m_0 the mass of the particle and G should be set to one:

$$\begin{aligned} \left(\frac{du}{d\varphi}\right)^2 &= \frac{2m_0m^2E_N}{J_N^2} - u^2 + \frac{2Gm_0^2m^2}{J_N^2}u \\ &= \frac{2m^2E_N}{J_N^2} - u^2 + \frac{2m^2}{J_N^2}u \\ &= \frac{m^2}{J_N^2}(2E_N + 2u) - u^2 \; . \end{aligned} \qquad (9)$$

and (Newton):

$$\frac{d^2u}{d\varphi^2} = -u + \frac{Gm_0^2m^2}{J_N^2}$$

$$= \ -u + \frac{m^2}{J_N^2} \ . \tag{10}$$

The u equation for Schwarzschild (5) with $\lambda = 1$ again:

$$\frac{d^2u}{d\varphi^2} = -u + 3u^2 + \frac{m^2}{J^2} \ . \tag{5}$$

The Newtonian solution derived previously:

$$u_0 = \frac{m^2}{J^2}(1 + e\cos\varphi) \ , \tag{11}$$

satisfies $\dot{u} = 0$ at $\varphi = 0$, and u has a *maximum* at $\varphi = 0$.

In (5) for u with $\lambda = 1$ we insert $u = u_0 + v$, where both u_0 and v are small, approximate

$$u^2 \approx u_0^2$$

to get

$$\frac{d^2v}{d\varphi^2} + v \approx 3\frac{m^4}{J^4}(1 + e\cos\varphi)^2 \ . \tag{12}$$

This can be integrated with e.g. MATHEMATICA to obtain

$$v(\varphi) \approx \frac{m^4}{J^4}\left[-(3+e^2)\cos\varphi + 3\left(1 + \frac{e^2}{2}\right) - \frac{e^2}{2}\cos 2\varphi + 3e\underline{\varphi\sin\varphi}\right] \ ,$$

where the free integration constants have been chosen so that $v(0) = v'(0) = 0$. Thus,

$$
\begin{aligned}
u(\varphi) &\approx \underbrace{\frac{m^2}{J^2}(1+e\cos\varphi)}_{u_0} \\
&\quad +\frac{m^4}{J^4}\left[-(3+e^2)\cos\varphi+3\left(1+\frac{e^2}{2}\right)-\frac{e^2}{2}\cos 2\varphi+3e\varphi\sin\varphi\right] .
\end{aligned}
$$

We seek the *perihelion*, the point of closest approach to the center, hence a maximum of u. Now

$$
\begin{aligned}
\partial_\varphi u(\varphi) &\approx -\frac{m^2}{J^2}e\sin\varphi \\
&\quad +\frac{m^4}{J^4}[(3+e^2)\sin\varphi+2\frac{e^2}{2}\sin 2\varphi \\
&\quad +3e\sin\varphi+3e\varphi\cos\varphi] .
\end{aligned}
$$

so this is indeed an extremum $\varphi=0$

$$
\partial_\varphi u = 0 \ ,
$$

and it is clear from the Newtonian expression (11) that this is a maximum of u for m/J small enough. The next maximum will be at $\varphi = 2\pi+\gamma$, with γ small:

$$
\begin{aligned}
0 &= \partial_\varphi u \\
&\approx -\frac{m^2}{J^2}e\sin\varphi+\frac{m^4}{J^4}[\underbrace{(3+e^2)\sin\varphi+e^2\sin 2\varphi+3e\sin\varphi}_{\approx\gamma\ll 1} \\
&\quad +3e\underbrace{\varphi}_{\approx 2\pi}\underbrace{\cos\varphi}_{\approx 1}] \\
&\approx -\frac{m^2}{J^2}e\gamma+6\frac{m^4 e\pi}{J^4} \ , \quad \text{since } \sin\varphi=\sin(2\pi+\gamma)\approx\gamma.
\end{aligned}
$$

We have thus obtained, in SI units, if $e \neq 0$,

$$\gamma \approx 6\frac{m^2\pi G}{J^2c^2} \ .$$

This is the *perihelion advance* predicted by general relativity. For Mercury this is sometimes called the *perihermion* advance, and equals about (exercice)

$$40'' \text{ per century } .$$

A more detailed calculation gives the observed value of $43''$ per century. This value was a puzzle to astronomers at the beginning of the twentieth century.

For the Taylor-Hulse pulsar the advance is around 4^o per year, which has to be corrected to account for the gravitational waves

§II.10: Geodetic precession

1. In Newtonian theory, consider a gyroscope rotating along a given direction, without any forces acting on it. Then the direction of rotation, and its velocity will remain unchanged. We can thus associate to a gyroscope a vector $\vec{s}$ such that

1. $\vec{s}$ points in the direction of rotation, and
2. $|\vec{s}|$ describes the speed of rotation.

Then we have

$$\frac{d\vec{s}}{dt} = 0 \ .$$

2. Let a gyroscope travel on a geodesic $\gamma(\tau)$ in Minkowski space-time, without any forces acting on it. In a Newtonian inertial frame the direction of rotation of the gyroscope will not change. It should then also be true that in a special relativistic inertial frame the direction of the gyroscope will not change either.

In inertial coordinates adapted to γ,

$$\gamma(\tau) = (\tau, \vec{0}) \ ,$$

the four-vector $s = (0, \vec{s})$ satisfies then, along γ, the set of equations

$$\frac{Ds}{d\tau} = 0 \ , \quad \eta(u, s) = 0 \ , \tag{13}$$

with η the Minkowski metric.

This is related to the notion of **non-rotating frames**.

3. One can invoke the correspondence principle to argue that

$$\frac{Ds}{d\tau} = 0 \ , \quad g(u, s) = 0 \ , \tag{14}$$

is the right general-relativistic equation to describe gyroscopes or, equivalently, **non-rotating local frames**.

Remember that

$$\frac{Ds^\mu}{d\tau} := \frac{ds^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} s^\alpha \dot{\gamma}^\beta \ .$$

§II.10.1: The parallel transport equation

Consider a geodesic $\gamma(\tau)$ with tangent vector $u \equiv \dot{\gamma} := d\gamma/d\tau$. As argued above, a gyroscope along γ can be described by a vector s orthogonal to u which is *parallel transported along* γ:

$$g(s, u) = 0 \ , \quad \frac{Ds}{d\tau} = 0 \ . \tag{15}$$

General facts about the **parallel transport equation**:

$$\frac{DX}{d\tau} = 0$$

Let X and Y be two vectors which are parallelly propagated along *any curve* $\gamma(\tau)$, not necessarily geodesic. Thus

$$\frac{DX}{d\tau} = 0 = \frac{DY}{d\tau} \ , \quad \text{with} \quad \frac{DX^\alpha}{d\tau} := \frac{dX^\alpha}{d\tau} + \Gamma^\alpha{}_{\mu\nu} X^\mu \frac{d\gamma^\nu}{d\tau} \ . \tag{16}$$

Now, for any vectors, regardless of parallel transport,

$$\frac{d}{d\tau}(g(X, Y)) \ = \ \frac{d}{d\tau}(g_{\mu\nu} X^\mu Y^\nu)$$

$$= \underbrace{\frac{dg_{\mu\nu}}{d\tau}}_{g_{\mu\nu,\sigma}\dot{\gamma}^\sigma} X^\mu Y^\nu + g_{\mu\nu}\frac{dX^\mu}{d\tau}Y^\nu + g_{\mu\nu}X^\mu\frac{dY^\nu}{d\tau} \ .$$

Since

$$0 = \nabla_\sigma g_{\mu\nu} = \partial_\sigma g_{\mu\nu} - \Gamma^\lambda_{\ \mu\sigma} g_{\lambda\nu} - \Gamma^\lambda_{\ \nu\sigma} g_{\lambda\mu} \ ,$$

equivalently,

$$\partial_\sigma g_{\mu\nu} = \Gamma^\lambda_{\ \mu\sigma} g_{\lambda\nu} + \Gamma^\lambda_{\ \nu\sigma} g_{\lambda\mu} \ ,$$

we obtain

$$\begin{aligned} \frac{d}{d\tau}(g(X,Y)) &= \underbrace{\frac{dg_{\mu\nu}}{d\tau}}_{(\Gamma^\lambda_{\ \mu\sigma} g_{\lambda\nu} + \Gamma^\lambda_{\ \nu\sigma} g_{\lambda\mu})\dot{\gamma}^\sigma} X^\mu Y^\nu + g_{\mu\nu}\frac{dX^\mu}{d\tau}Y^\nu + g_{\mu\nu}X^\mu\frac{dY^\nu}{d\tau} \\ &= g(\frac{DX}{d\tau}, Y) + g(X, \frac{DY}{d\tau}) \ . \end{aligned}$$

Summarising:

$$\boxed{\tfrac{d}{d\tau}(g(X,Y)) = g(\tfrac{DX}{d\tau}, Y) + g(X, \tfrac{DY}{d\tau}) \ .}$$

For parallel-transported vectors,

$$\boxed{\tfrac{d}{d\tau}(g(X,Y)) = 0 \ .}$$

Thus, the angle between the vectors is preserved along the curve.

In particular if $X \perp Y$ at one point, then it will be so along the whole curve.

Similarly if $g(X,X) = \pm 1$ at one point, it will be so along the whole curve.

Paralelly transporting an ON basis e_a of the tangent space $a = 0, \ldots, n$ at some point of γ we will obtain an ON basis of $T\mathscr{M}$ at any point lying on the image of γ.

If γ is an affinely parameterised geodesic, then the tangent vector u is parallel along γ. So, for a gyroscope, the first condition

$$g(s, u) = 0$$

in (15) is consistent with the evolution equations for s and u. Equivalently, if $g(s, u)$ vanishes at one point of the geodesic, it will vanish everywhere.