Geodesics in Schwarzschild, Reminders:

$J \neq 0$,

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{m^2E^2}{J^2} - u^2(1-2u) - \frac{\lambda m^2(1-2u)}{J^2}\ , \qquad (1)$$

where $\lambda = 0$ for null geodesics, and $\lambda = 1$ for timelike ones.

$$\frac{d^2u}{d\varphi^2} = -u + 3u^2 + \frac{\lambda m^2}{J^2}\ . \qquad (2)$$

$$\frac{dt}{ds} = \frac{E}{1 - \frac{2m}{r}}\ . \qquad (3)$$

$$\frac{d\varphi}{ds} = \frac{J}{r^2}\ . \qquad (4)$$

Stability of circular geodesics

Consider a geodesic which is a small perturbation of the fixed-radius geodesic $u(s) = u_0$, where

$$0 = \frac{d^2u_0}{d\varphi^2} = -u_0 + 3u_0^2 + \frac{\lambda m^2}{J^2}\ . \qquad (5)$$

We set $u = u_0 + \delta u$, where δu is assumed to be small.
Linearizing (2) at u_0 (so $u^2 \approx u_0^2 + 2u_0\delta u$)

$$\frac{d^2\delta u}{d\varphi^2} \approx -\delta u + 6u_0\delta u = (6u_0 - 1)\delta u\ .$$

1. When $6u_0 > 1$ (equivalently, $r_0 < 6m$) the solutions are linear combinations of $e^{\pm\sqrt{6u_0-1}\varphi}$ and thus linearization-unstable.

2. For $6u_0 < 1$ (equivalently, $r_0 > 6m$) the solutions are linear combinations of $\sin(\sqrt{|6u_0 - 1|}\varphi)$ and $\cos(\sqrt{|6u_0 - 1|}\varphi)$, and thus linearization-stable.

So:

circular timelike geodesics are stable if and only if $r > 6m$.

The orbits $r = 6m$ are called *innermost stable circular orbits* (ISCO) in the astrophysical literature.

Parenthesis: the Newtonian many body problem

§II.8 Photons

In special relativity photons move along straight lines with null tangent $\eta(\dot\gamma,\dot\gamma) = 0$; these are affinely parameterized geodesics of the Minkowski metric η. In view of the correspondence principle we require that

test photons in general relativity move along null geodesics.

Here, a *test photon* is a photon, the gravitational field of which can be ignored at the scale at which experiments are carried out.

§II.8.1: Circular null geodesics

Recall (1) and (2)

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{m^2E^2}{J^2} - u^2(1-2u) - \frac{\lambda m^2(1-2u)}{J^2}\,, \qquad (1)$$

$$\frac{d^2u}{d\varphi^2} = -u + 3u^2 + \frac{\lambda m^2}{J^2}\,. \qquad (2)$$

Set $\lambda = 0$ in (1) and (2)

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{m^2E^2}{J^2} - u^2(1-2u)\,,$$

$$\frac{d^2u}{d\varphi^2} = -u + 3u^2\,. \qquad (6)$$

so $r = \text{const}$ if and only if $u = \text{const}$ if and only if

$$u = 3 \quad \Longleftrightarrow \quad r = 3m$$

(PS3, Q2: $r = 2m$, $\theta = \text{const}$, $\varphi = \text{const}$ are also null geodesics but they are not in this class, why?)

Simple algebra shows then that the curves

$$s \mapsto \gamma_\pm(s) = (t = s,\ r = 3m,\ \theta = \pi/2,\ \varphi = \pm 3^{\frac{3}{2}} m^{-1} s)\,,$$

are null geodesics spiraling on the timelike cylinder $\{r = 3m\}$.

§II.8.2 Gravitational redshift

In this section we wish to derive the frequency shift along radial null geodesics.

Let a wave of light with frequency ω_1 be emitted radially at r_1, and let

$$\Delta s_1 := \frac{2\pi}{\omega_1}$$

be the proper time between two consecutive maxima of the wave, which travels outwards on radial null geodesics.

The tangent vector to such geodesics has zero length, and θ and ϕ are constant,

$$0 = -g(\dot{\gamma}, \dot{\gamma}) = \left(1 - \frac{2m}{r}\right) (\frac{dt}{ds})^2 - \frac{(\frac{dr}{ds})^2}{1 - 2m/r} = 0\,,$$

leading to

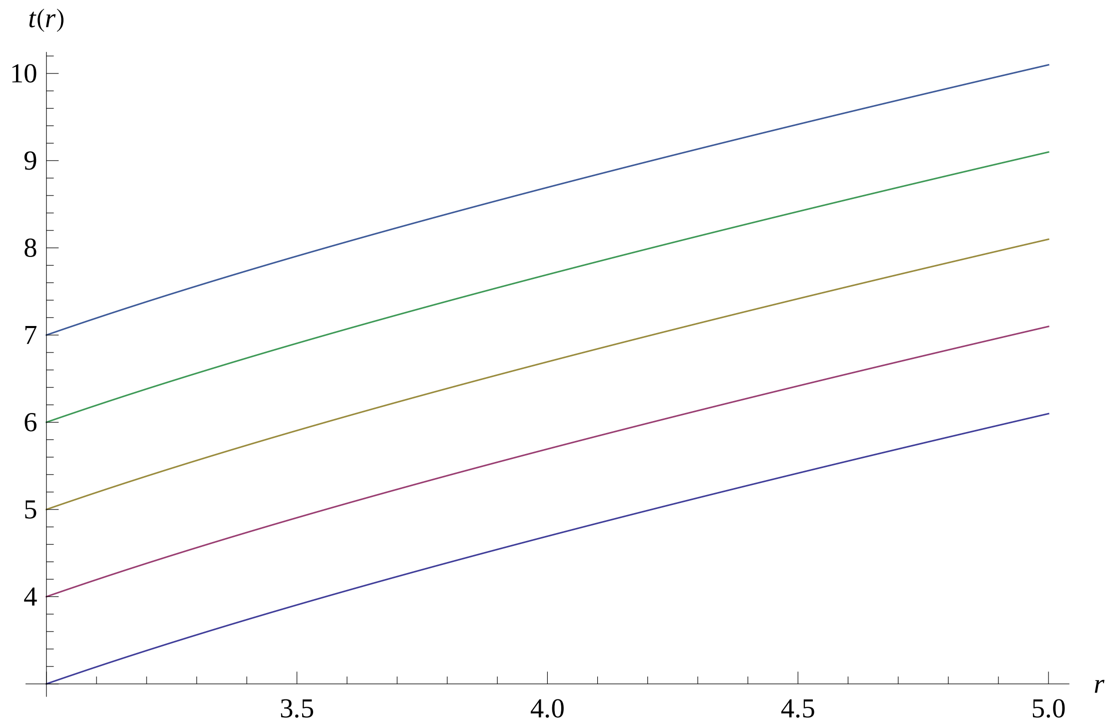
$$\frac{dt}{dr} = \frac{1}{1 - 2m/r} = \frac{r}{r - 2m}\,. \tag{7}$$

By integrating (7) we find that the coordinate time t_1 at which the crest of the wave leaves r_1 is related to the coordinate time t_2 at which it arrives at r_2 by

$$t_2 - t_1 = \int_{r_1}^{r_2} \frac{r\ dr}{r - 2m}\,. \tag{8}$$

The right-hand side is of (8) is independent of t_1, so:

the coordinate time interval Δt_1 between the emission times of two successive crests at r_1 = the coordinate time interval Δt_2 between their observations at r_2.



To determine what happens with frequency, we have to relate coordinate time to proper time of observers.

Recall that the four-velocity vector

$$u \equiv u^\mu \partial_\mu := \frac{dx^\mu}{ds}\partial_\mu$$

of a stationary observer at coordinate radius r takes the form

$$u = \frac{1}{\sqrt{1-\frac{2m}{r}}}\partial_t = \frac{dt}{ds}\partial_t \quad \Longleftrightarrow \quad \frac{dt}{ds} = \frac{1}{\sqrt{1-\frac{2m}{r}}}\,. \tag{9}$$

Since the emitter of light has been assumed to be stationary, (9) shows that the proper time interval Δs_1 is related to the coordinate time interval Δt_1 by

$$\Delta t_1 = \frac{dt}{ds}\Delta s_1 = \frac{1}{\sqrt{1-\frac{2m}{r_1}}}\Delta s_1\ ,$$

with a similar formula relating Δt_2 with Δs_2.

$$\Delta t_2 = \frac{dt}{ds}\Delta s_2 = \frac{1}{\sqrt{1-\frac{2m}{r_2}}}\Delta s_2 \ ,$$

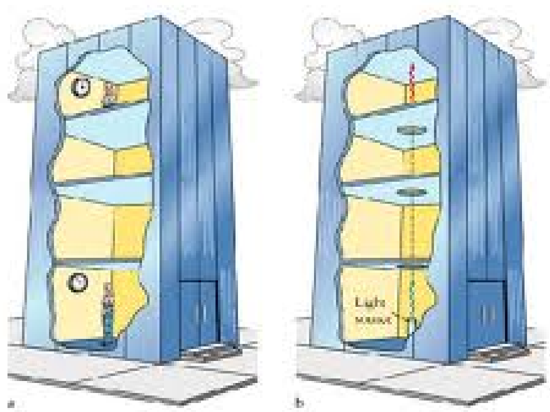
As we have seen that $\Delta t_1 = \Delta t_2$, we obtain

$$\frac{\Delta s_1}{\Delta s_2} = \frac{\sqrt{1-2m/r_1}}{\sqrt{1-2m/r_2}} \, . \tag{10}$$

Subsequently,

$$\omega_2 = \frac{\sqrt{1-\frac{2m}{r_1}}}{\sqrt{1-\frac{2m}{r_2}}}\omega_1 \ . \tag{11}$$

This is the *gravitational redshift formula*, as observed e.g. in the Pound-Rebka experiment, see e.g. http://en.wikipedia.org/wiki/Pound-Rebka_experiment. The detailed de-



scription of the experiment from http://en.wikipedia.org/wiki/Pound%E2%80%93Rebka_experiment:

A solid sample containing iron (57Fe) emitting gamma rays was placed in the center of a loudspeaker cone which was placed near the roof of the building. Another sample containing 57Fe was placed in the basement. The distance

between this source and absorber was 22.5 meters (73.8 ft). The gamma rays traveled through a Mylar bag filled with helium to minimize scattering of the gamma rays. A scintillation counter was placed below the receiving 57Fe sample to detect the gamma rays that were not absorbed by the receiving sample. By vibrating the speaker cone the gamma ray source moved with varying speed, thus creating varying Doppler shifts. When the Doppler shift canceled out the gravitational blueshift, the receiving sample absorbed gamma rays and the number of gamma rays detected by the scintillation counter dropped accordingly. The variation in absorption could be correlated with the phase of the speaker vibration, hence with the speed of the emitting sample and therefore the doppler shift. To compensate for possible systematic errors, Pound and Rebka varied the speaker frequency between 10 Hz and 50 Hz, interchanged the source and absorber-detector, and used different speakers (ferroelectric and moving coil magnetic transducer). The effect to measure is rather small,

$$\frac{\delta\omega}{\omega} = 2.5 \times 10^{-15} \ ,$$

which was confirmed to about 10%.

$$\omega_{(2)} = \frac{\sqrt{1 - \frac{2m}{r_{(1)}}}}{\sqrt{1 - \frac{2m}{r_{(2)}}}}\omega_{(1)} \ . \qquad (11)$$

The frequency observed by an observer at infinity will be smaller than the energy emitted at any finite radius.

If $r_{(2)} > r_{(1)}$, then the observed spectrum will be shifted to the red, by a frequency-independent multiplicative factor, as compared to the emitted one.

As another application of (11), imagine that you send a beacon emitting at constant frequency towards a black-hole. You will see its frequency shifting away towards the red as the beacon approaches the event horizon $r = 2m$, with the frequency tending to zero as the event horizon is approached.

§II.8.3 Weak field light bending

For null geodesics (2) reads

$$\frac{d^2u}{d\varphi^2} = -u + 3u^2 \ . \tag{12}$$

For u very small or, equivalently, for r large as compared to m, an excellent approximation is obtained by neglecting the quadratic term, leading to

$$u_0 = \alpha \cos(\varphi - \varphi_0) \ ,$$

for some small constant $\alpha \neq 0$ (otherwise $r = \infty$). By a redefinition of φ we can always achieve $\varphi_0 = 0$. Equivalently,

$$\alpha r \cos\varphi = m \qquad \Longleftrightarrow \qquad x = m/\alpha \ ,$$

straight line in the (x, y) plane passing through $(d, 0)$, where d is the distance of closest approach to the origin, with

$$\alpha = \frac{m}{d} \ .$$

We can calculate the leading order correction to this by writing $u = \alpha\cos\varphi + v(\varphi)$, where $v = O(\alpha^2)$ is small. Inserting into (12) and neglecting terms which are $O(\alpha^3)$ one obtains

$$v'' + v = 3\alpha^2\cos^2\varphi \ .$$

This is easily (e.g. with MATHEMATICA) integrated to give

$$v = A\cos\varphi + B\sin\varphi + \alpha^2(1 + \sin^2\varphi) \ .$$

We choose A and B so that at $\varphi = 0$ the initial data for the orbit coincide with those for the unperturbed one,

$$u(0) = \frac{m}{d} =: \alpha = u_0(0) \ , \quad u'(0) = 0 = u_0'(0) \ ,$$

Now,

$$u'(0) = (-A\sin\varphi + B\cos\varphi + 2\alpha^2\sin\varphi\cos\varphi)|_{\varphi=0} = B = 0 \ ,$$

$$\begin{aligned} v(0) &= (A\cos\varphi + \underbrace{B}_{0}\sin\varphi + \alpha^2(1+\sin^2\varphi))|_{\varphi=0} \\ &= A + \alpha^2 = 0 \quad \Longrightarrow \quad A = -\alpha^2 \ . \end{aligned}$$

Finally,

$$\begin{aligned} u &= (\alpha - \alpha^2)\cos\varphi + \alpha^2(1+\sin^2\varphi) + O(\alpha^3) \\ &\approx (\alpha - \alpha^2)\cos\varphi + \alpha^2(1+\sin^2\varphi) \ . \end{aligned}$$

recall that

$$\alpha = \frac{m}{d} \ ,$$

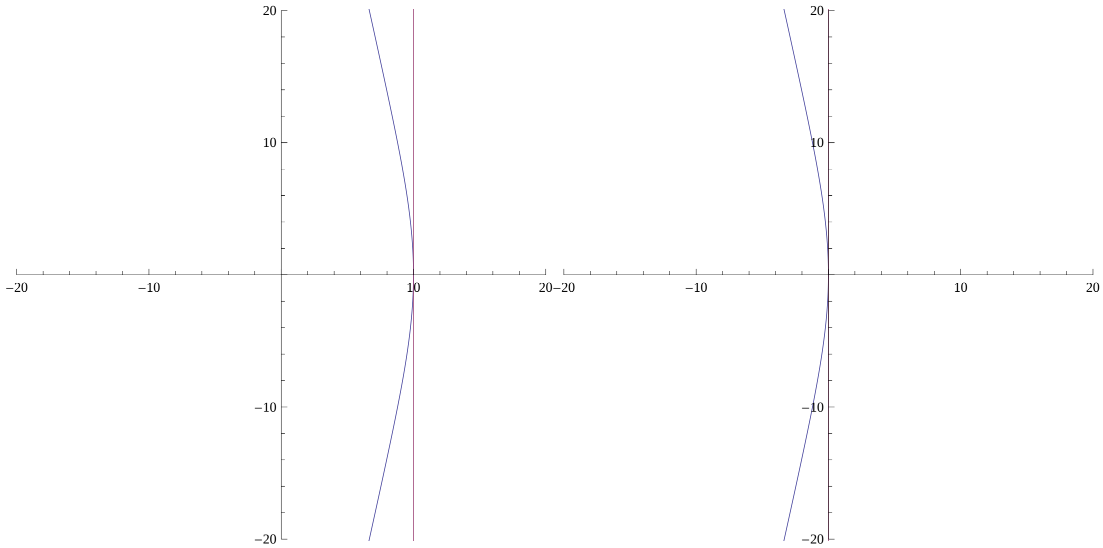
and that the "Newtonian" orbit $u_0(\varphi) = \alpha\cos\varphi$ was a straight line.

Note that d is the distance of the closest approach of the orbit to the origin $r = 0$ for α small: indeed,

$$\begin{aligned} u' &\approx -(\alpha-\alpha^2)\sin\varphi + 2\alpha^2\sin\varphi\cos\varphi \\ &= \big(-(\alpha-\alpha^2) + 2\alpha^2\cos\varphi\big)\sin\varphi \\ &= \alpha\left(-1 + \underbrace{\alpha + 2\alpha\cos\varphi}_{\ll 1}\right)\sin\varphi \\ &\neq 0 \qquad \text{for } \varphi \in (-\pi,\pi)\setminus\{0\} \ . \end{aligned}$$

Thus, up to α^3 corrections, u has only one extremum, at $\varphi = 0$, which is a maximum, and so r has a minimum at $\varphi = 0$.

By definition of u, we have $r \to \infty$ if and only if $u \to 0$; for $u = u_0 = \alpha\cos\varphi$ this corresponds to $\varphi \to \pm\pi/2$.



However, the corrected orbit u will reach zero at angles

$$\varphi_\alpha = \pm(\pi/2 + \gamma_\alpha) ,$$

slightly larger in modulus than $\pi/2$:

$$u \;\approx\; (\alpha - \alpha^2)\cos\varphi + \alpha^2(1 + \sin^2\varphi) .$$

$$\begin{aligned} 0 &\approx \frac{u(\varphi_\alpha)}{\alpha} \\ &= (1-\alpha)\cos\varphi_\alpha + \alpha(1+\sin^2\varphi_\alpha) \\ &= (1-\alpha)\cos(\pi/2+\gamma_\alpha) + \alpha(1+\sin^2(\pi/2+\gamma_\alpha)) \\ &= -(1-\alpha)\underbrace{\sin\gamma_\alpha}_{\approx\gamma_\alpha} + \alpha(1+\underbrace{\cos^2\gamma_\alpha}_{\approx 1}) \\ &\approx -\underbrace{(1-\alpha)}_{\approx 1}\gamma_\alpha + 2\alpha \\ &\approx -\gamma_\alpha + 2\alpha , \end{aligned}$$

so

$$\gamma_\alpha = 2\alpha + O(\alpha^2) .$$

The total bending of the orbit is 4α, giving the final SI formula for the angle deflection

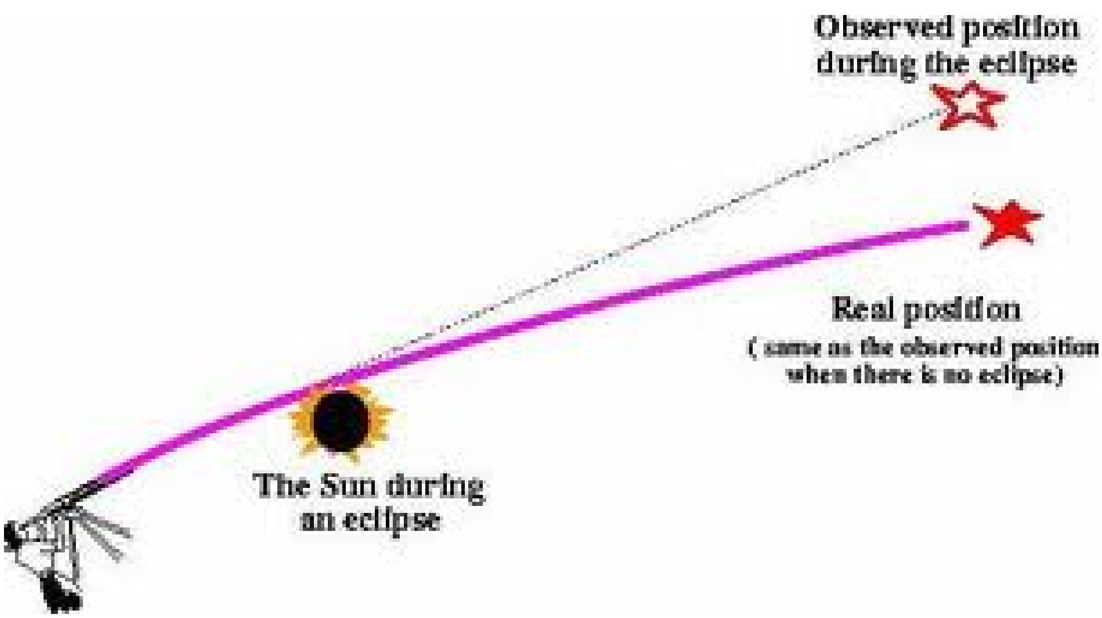
$$\frac{4mG}{dc^2},$$

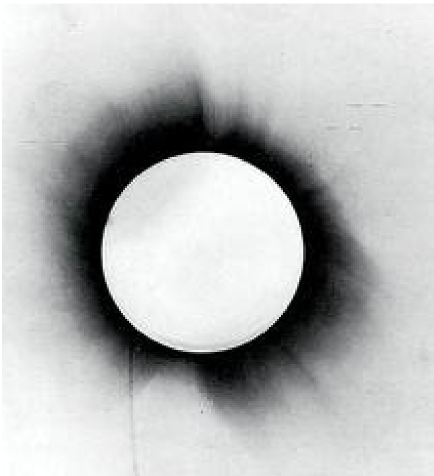
recall that d is the distance of closest approach to the center.

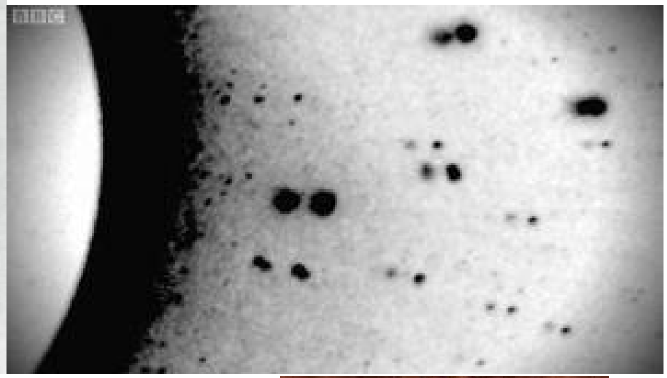
For a light ray just grazing the surface of the sun, so that $m = M_{☼}$, $d = r_{♁}$, one obtains a deflection of

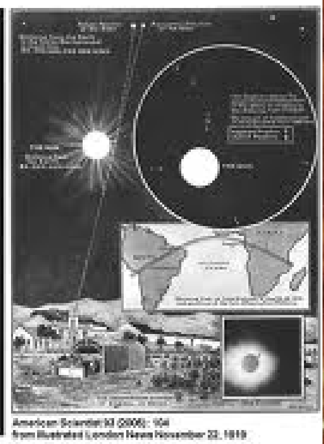
$$10^{-5} \text{ radians or } 2''.$$

Observed (?? discarded observations at Sobral, Brazil; telescope problem ??) by Eddington during the 1919 eclipse expedition to the Principe Island, off the cost of Africa, by comparing photographs of the star field near the sun during an eclipse with a photograph of the same star field when the sun was not interfering.





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EINSTEIN AND EDDINGTON