1 By using Taylor expansions in time to second order of all relevant quantities, show that in a FRW metric the space-distance d is related to the red-shift factor z by the formula

$$d \approx \frac{1}{H_0}\left(z - \frac{1+q_0}{2}z^2\right),$$

where H_0 is the cosmological constant and q_0 the deceleration parameter.

2 Consider the Friedman–Robertson–Walker metric

$$g = -dt^2 + R(t)^2\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right],$$

and the corresponding Friedman equation

$$\left(\frac{dR}{dt}\right)^2 - \frac{8\pi}{3}\rho(t)R(t)^2 = -k.$$

Fix a reference time $t = t_0$. Assuming that the matter content consists of a pressure-free gas, show that the energy density and scale factor obey the law

$$\rho(t)R(t)^3 = \rho(t_0)R(t_0)^3.$$

Define the conformal time coordinate η such that

$$\frac{dt}{R(t)} = d\eta,$$

where we impose that $\eta(t=0) = 0$. Show that the metric takes the conformally flat form

$$g = R(t(\eta))^2\left[-d\eta^2 + \frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right].$$

Show that the Friedman equation may now be written in the form

$$\left(\frac{da}{d\eta}\right)^2 = \frac{8\pi}{3}\rho(t_0)R(t_0)^3 R(\eta) - kR(\eta)^2.$$

Define the Hubble constant (at $t = t_0$), the critical energy density and the parameter Ω by

$$H_0 := \left.\frac{dR/dt}{R}\right|_{t=t_0}, \qquad \rho_{crit} := \frac{3}{8\pi}H_0^2, \qquad \Omega := \frac{\rho(t_0)}{\rho_{crit}}.$$

Treating the cases $k = 1, k = -1$ and $k = 0$ separately, show that the solutions of the Friedman equation satisfying $R(\eta = 0) = 0$ (and $R(\eta) \geq 0$ for sufficiently small η) take the form

$$R(\eta) = \frac{\Omega}{2H_0(\Omega-1)^{3/2}}\left[1-\cos\eta\right], \qquad k = 1,$$

$$R(\eta) = \frac{1}{4}H_0^2 R(t_0)^3\eta^2, \qquad k = 0,$$

$$R(\eta) = \frac{\Omega}{2H_0(1-\Omega)^{3/2}}\left[\cosh\eta - 1\right], \qquad k = -1.$$

and, therefore,

$$
\begin{aligned}
t(\eta) &= \frac{\Omega}{2H_0\,(\Omega-1)^{3/2}}\,[\eta+\sin\eta]\,, && k=1,\\
t(\eta) &= \frac{1}{12}H_0^2 R(t_0)^3\eta^3, && k=0,\\
t(\eta) &= \frac{\Omega}{2H_0\,(1-\Omega)^{3/2}}\,[\sinh\eta-\eta]\,, && k=-1.
\end{aligned}
$$

For what length of time (i.e. t) does the $k = 1$ universe exist? Show that, for the $k = 0$ universe, we have

$$
\frac{R(t)}{R(t_0)} = \left(\frac{3}{2}H_0 t\right)^{2/3} .
$$

3 Repeat Question 2 for the radiation filled universe, where one has $p(t) = \frac{1}{3}\rho(t)$.