1 Find the flow of the following vector fields:

i. $X = x\partial_x + y\partial_y$ on $\mathbb{R}^2$

ii. $X = x\partial_y - y\partial_x$ on $\mathbb{R}^3$

iii. Let $M = \{a = (a_{ij})\ ,\ a_{ij} \in \mathbb{R}\}$ be the set of all $n \times n$ matrices, and let $X(a) = a_{ij}\frac{\partial}{\partial a_{ij}}$.

2 Given a one-form, α, we define its exterior derivative, $d\alpha$, as

$$d\alpha_{ab} = \partial_a\alpha_b - \partial_b\alpha_a.$$

Show that $d\alpha$ is a tensor.

Show that, acting on arbitrary vector fields X, Y, we have

$$d\alpha(X,Y) = \mathcal{L}_X(\alpha(Y)) - \mathcal{L}_Y(\alpha(X)) - \alpha([X,Y]).$$

(Note that this allows us to define the exterior derivative of a one-form in terms of the Lie derivative.) Also, show that for arbitrary vector fields X, Y, we have

$$(\mathcal{L}_X\alpha)(Y) = d\alpha(X,Y) + Y(\alpha(X)).$$

Recall that if f is a function, then $df = \partial_i f dx^i$. Show that $d(df) = 0$, and

$$\mathcal{L}_X(df) = d(\mathcal{L}_X f)\ .$$

3 The Heisenberg group is the group of matrices

$$G := \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x,y,z \in \mathbb{R} \right\},$$

with the group operation given by matrix multiplication. Given fixed $g = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in G$, we define the operation of left-multiplication $L_g \colon G \to G$ by $h \mapsto gh$. Calculate the image of $h = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$ under the map L_g, and the push-forward map $\left(L_g\right)_*$ at h. A vector field on G is *left-invariant* if $\left(L_g\right)_* v = v$ for all $g \in G$. Show that the vector fields

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_y, \qquad \mathbf{v}_3 = \partial_z + x\partial_y$$

are left-invariant vector fields on G. Calculate the flows of the vector fields $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ on G (i.e. the integral curves of the vector fields starting at a fixed point $g_0 \in G$).