1 QI. The Lane-Emden equation

Recall from lectures that, for a Newtonian static fluid, p and ρ satisfy the equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi\rho.$$

Assume that $\rho = \rho_c\theta(r)^n$ and $p = p_c\theta(r)^{n+1}$, where n, ρ_c, p_c are constants, and θ is a function to be determined. Check that

$$p = K\rho^{\frac{n+1}{n}},$$

for a constant K that you will determine. Show that the function θ satisfies the differential equation

$$\frac{K(n+1)}{4\pi}\rho_c^{\frac{1}{n}-1}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n.$$

Let $r_n > 0$ be defined by

$$r_n^2 = \frac{K(n+1)}{4\pi}\rho_c^{\frac{1}{n}-1},$$

and define a new variable ξ by $r = r_n\xi$. Deduce that

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n.$$

Solve this equation with boundary conditions $\theta(0) = 1$, $\theta'(0) = 0$ in the case $n = 0$. In the case $n = 1$, let $\theta_1(\xi) = \theta(\xi)/\xi$, and solve for $\theta_1(\xi)$ with the same boundary conditions. Finally, verify that the function

$$\theta_5(\xi) = \frac{1}{\left(1 + \frac{1}{3}\xi^2\right)^{1/2}}$$

is a solution in the case $n = 5$.

2 QII. Lie derivative

Given a vector field X, the *Lie derivative* $\mathcal{L}_X$ is an operation on tensors, defined as follows:

i. For a function f, one sets $\mathcal{L}_X f := X(f)$.

ii. For a vector field Y, one sets $\mathcal{L}_X Y := [X, Y]$, the Lie bracket.

iii. For a one form α, $\mathcal{L}_X\alpha$ is defined by imposing the Leibniz rule written backwards:

$$(\mathcal{L}_X\alpha)(Y) := \mathcal{L}_X(\alpha(Y)) - \alpha(\mathcal{L}_X Y)\ .$$

Q II.1: a) Why is this the same as the Leibniz rule? *[Hint: write this equation using indices.]* b) Check that $\mathcal{L}_X\alpha$ is a tensor. *[Hint: check that the right-hand-side is linear under multiplication of Y by a function.]*

Q II.2: Show that

$$(\mathcal{L}_X\alpha)_a = X^b\partial_b\alpha_a + \alpha_b\partial_a X^b \ .$$

iv. For tensor products the Lie derivative is defined again by imposing linearity under addition together with the Leibniz rule:

$$\mathcal{L}_X(\alpha\otimes\beta) = (\mathcal{L}_X\alpha)\otimes\beta + \alpha\otimes\mathcal{L}_X\beta \ .$$

Since a general tensor A is sum of tensor products,

$$A = A^{a_1\ldots a_p}{}_{b_1\ldots b_q}\partial_{a_1}\otimes\ldots\partial_{a_p}\otimes dx^{b_1}\otimes\ldots\otimes dx^{a_p} \ ,$$

requiring linearity with respect to addition of tensors gives thus a definition of Lie derivative for any tensor.

Q II.3: Show that

$$\mathcal{L}_X T^a{}_b = X^c\partial_c T^a{}_b - T^c{}_b\partial_c X^a + T^a{}_c\partial_b X^c \ ,$$

Q II.4: Can you see the general formula for the Lie derivative $\mathcal{L}_X A^{a_1\ldots a_p}{}_{b_1\ldots b_q}$?