1 Calculate the field of unit normals and the induced metric for

i. S^2 included in a flat $\mathbb{R}^3$ (use polar coordinates on S^2)

ii. S^2 viewed as a sphere of constant radius in Schwarzschild (use polar coordinates on S^2)

iii. $\mathcal{S} = \{x^0 = \sqrt{1 + r^2}\}$ in four dimensional Minkowski; use the cartesian space coordinates x^i as coordinates on $\mathcal{S}$

iv. $\mathcal{S}' = \{r = \sqrt{1 + t^2}\}$ in four dimensional Minkowski; use t and polar coordinates as coordinates on $\mathcal{S}'$

2 Find a function f so that the metric induced on the hypersurface $\{t = f(r)\}$ in Minkowski space-time is flat.

3 A vector field X^μ is called *Killing* if

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0 \ .$$

i. Check that ∂_t and ∂_φ are Killing vectors both in the Minkowski metric and in the Schwarzschild metric.

ii. Show that if γ is a geodesic and X is Killing, then $g(\gamma, X)$ is constant along γ.

iii. Show that if $\nabla_\mu T^\mu{}_\nu = 0$ and X is Killing then the vector field $J^\mu := T^\mu{}_\nu X^\nu$ has vanishing divergence, $\nabla_\mu J^\mu = 0$.

4 Let φ satisfy the wave equation, $\Box\varphi := \nabla^\mu \nabla_\mu \varphi = 0$. Set

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}\nabla^\alpha\varphi\partial_\alpha\varphi g_{\mu\nu} \ .$$

Show that $\nabla_\mu T^\mu{}_\nu = 0$.

When $X = \partial_t$ is a Killing vector, the constant of motion associated with the current J of question 3iii is called the total energy E of the field. Give an explicit expression for E for the surface $\{t = 0\}$ in the Minkowski space-time, and in the Schwarzschild space-time.