1 Let $g_{\mu\nu} = \eta_{\mu\nu} + A_{\mu\nu}\cos(k_\alpha x^\alpha)$ be the metric of a linearized plane wave in the gauge $A_{\mu\nu}k^\mu = 0 = A_{\mu\nu}U^\mu = A_\mu{}^\mu$, where $k^\mu\partial_\mu = \omega(\partial_t + \partial_z)$, $U^\mu\partial_\mu = \partial_t$. Using the variational principle for geodesics, write down the geodesic equations. Introducing the variables $u = t - z$, $v = t + z$, find the geodesics.

[*Hint*: You should find that there are two classes of geodesics. The first class have u = constant, in which case you should be able to integrate the geodesic equations completely. For the geodesics with u non-constant, use u as a parameter along the geodesics, and solve the equations neglecting terms of order A^2.]

2 Consider a body of mass m_0 and angular momentum J_N moving in an elliptic, Kepler orbit of eccentricity e in the field of a central mass of mass m. (See Problem Sheet 5, question 2.) Assume that, in Cartesian coordinates, this motion takes place in the $z = 0$ plane. Show that, at distance R, and time t, in the linearized approximation, the quadrupole moment of the system takes the form

$$q_{ij} = 3m_0 r(\varphi(t))^2 \begin{pmatrix} \cos^2\varphi(t) & \sin\varphi(t)\cos\varphi(t) & 0 \\ \sin\varphi(t)\cos\varphi(t) & \sin^2\varphi(t) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Using the explicit form of $r(\varphi(t))$ derived in Problem Sheet 5, question 2, derive an expression for the linearised metric perturbation $\overline{h}_{xx}$, to first order in the eccentricity e.