1 Null Geodesics in Schwarzschild Let $\gamma$ be an affinely parameterized null geodesic for the Schwarzschild metric with $J \neq 0$ lying in the equatorial plane $\theta = \pi/2$.

Q1. Show that along $\gamma$, both for $u > 1/2$ and for $u < 1/2$, we have

$$p^2 = 2u^3 - u^2 + \alpha^2 \ , \tag{1}$$

where $\alpha$ is a constant and

$$u = \frac{m}{r} \ , \quad p = \frac{du}{d\varphi} \ .$$

Q2. For the case $\alpha = 0$ show that we obtain an autonomous two-dimensional system

$$\frac{du}{d\phi} = p \ , \quad \frac{dp}{d\phi} = 3u^2 - u \ .$$

Find and classify the critical points. Sketch the trajectories in the $(u, p)$ phase-plane.

Q3. For the case $\alpha = 0$ and $u > 1/2$ show that the geodesic has an equation of the form

$$r = 2m\cos^2\left(\frac{\varphi - \varphi_0}{2}\right) \ , \tag{2}$$

with $t = t(\varphi)$ that you should determine.

2 Timelike geodesics in Schwarzschild Let $\gamma$ be a timelike geodesic for the Schwarzschild metric parameterized by proper time and lying in the equatorial plane $\theta = \pi/2$.

Q1. Show that

$$\frac{E^2 - \dot{r}^2}{1 - \frac{2m}{r}} - \frac{J^2}{r^2} = 1 \ .$$

Q2. Deduce that if $E = 1$ and $J = 4m$ then

$$\frac{\sqrt{r} - 2\sqrt{m}}{\sqrt{r} + 2\sqrt{m}} = Ae^{\epsilon\varphi/\sqrt{2}} \ ,$$

where $\epsilon = \pm 1$ and $A$ is a constant. Describe the orbit that starts at $\varphi = 0$ in each of the cases (i) $A = 0$, (ii) $A = 1$, $\epsilon = -1$, (iii) $r(0) = 3m$, $\epsilon = -1$.