(1) Show that the tensor field $g_{\mu\nu}(x) = \eta_{\mu\nu}$ is invariant under Poincaré transformations, i.e. $x^\mu \mapsto \bar{x}^\mu = L^\mu{}_\nu x^\nu + c^\mu$, where $L^\mu{}_\nu$ is a constant matrix subject to $L^\mu{}_\rho L^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma}$ and c^μ are constants. Why does $L^\mu{}_\nu$ have 6 degrees of freedom in $n = 4$?

(2) Show that the vector field v^A $(A = 1, 2)$ given by $v^A \partial_A = -x^2 \partial_1 + x^1 \partial_2$ is invariant under rotations in $\mathbb{R}^2$.

(3) Show that the $(1, 1)$ - tensor field $t^\mu{}_\nu(x) = \delta^\mu{}_\nu$ is invariant under general transformations.

(4) Show that (any) contraction of a (p, q) - tensor results in a $(p-1, q-1)$ - tensor.

(5) Show that every covector field $\omega_\mu(x)$ is a finite combination of terms of the form $\phi\, \partial_\mu \psi$, where ϕ, ψ are scalar fields. Hint: Consider for the ψ's the functions x^μ with $\mu = 1, 2..., n$.

(6) Prove that, for a torsion free connection ∇_μ,

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)(\phi\, \omega_\rho) = \phi\, (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\, \omega_\rho \tag{0.1}$$

(7) Show that

$$\nabla_{[\mu} \nabla_\nu (\phi \nabla_{\rho]} \psi) = 0 \tag{0.2}$$

(8) Show that the geodesic equation for a curve $x^\mu = z^\mu(\lambda)$ is the Euler-Lagrange equation for

$$S[z] = \int (\dot{z}, \dot{z})\, d\lambda \tag{0.3}$$

(9) Suppose two metrics, $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$, are related by

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \tag{0.4}$$

Show that the corresponding Christoffel symbols are related by

$$\bar{\Gamma}^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} + \frac{1}{2}\, \bar{g}^{\mu\sigma}\, (2\nabla_{(\nu} h_{\rho)\sigma} - \nabla_\sigma h_{\nu\rho}) \tag{0.5}$$

where ∇_μ is the covariant derivative associated with the metric $g_{\mu\nu}$ and $\bar{g}^{\mu\rho} \bar{g}_{\rho\nu} = \delta^\mu{}_\nu$.

(10) Let ∇_μ be any covariant derivative, not necessarily associated with some metric. Show that the expressions

$$(\mathcal{L}_\xi \eta)^\mu = \xi^\nu \nabla_\nu \eta^\mu - \eta^\nu \nabla_\nu \xi^\mu \tag{0.6}$$

$$(\mathcal{L}_\xi \omega)_\mu = \xi^\nu \nabla_\nu \omega_\mu + \omega_\nu \nabla_\mu \xi^\nu \tag{0.7}$$

$$(\mathcal{L}_\xi t)_{\mu\nu} = \xi^\rho \nabla_\rho t_{\mu\nu} + 2t_{\rho(\mu} \nabla_{\nu)} \xi^\rho, \qquad t_{\mu\nu} = t_{(\mu\nu)} \tag{0.8}$$

$$(d\omega)_{\mu\nu\lambda} = \nabla_{[\mu} \omega_{\nu\lambda]}\,, \qquad \omega_{\mu\nu} = \omega_{[\mu\nu]} \tag{0.9}$$

are independent of the chosen connection. Can you think of any other differential operators having this property?

(11) Let $\delta R_{\mu\nu}[h]$ be the Ricci tensor of a weak gravitational field computed in the lecture course. Prove that $\delta R_{\mu\nu}[h]$ vanishes for any h of the form $h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$.

(12) Let $F^\mu(\epsilon, x)$ a 1-parameter family of transformations with $F^\mu(0, x) = x^\mu$, all of which are symmetries of the metric $g_{\mu\nu}$. Show that this implies

$$(\mathcal{L}_\xi g)_{\mu\nu} = \xi^\rho\, \partial_\rho g_{\mu\nu} + g_{\rho\nu}\, \partial_\mu \xi^\rho + g_{\mu\rho}\, \partial_\nu \xi^\rho = 0\,, \tag{0.10}$$

where $\xi^\mu(x) = \frac{d}{d\epsilon} F^\mu(\epsilon, x)|_{\epsilon=0}$.

(13) Consider the line element given by

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2\,, \qquad 0 < \theta < \pi\,,\ 0 \le \phi \le 2\pi \tag{0.11}$$

This has $\xi = \frac{\partial}{\partial\phi}$ as Killing vector. (Is there a computation-free argument for this? Hint: see Ex.(12)). Find two 'more' Killing vectors of ds^2. Hint: consider the vector fields $\eta = x^1\partial_2 - x^2\partial_1$, a.s.o. on $\mathbb{R}^3$ and introduce polar coordinates. (Remark: 'more' means the three resulting Killing vectors $\xi_{(i)}$ are independent in the sense that $\sum_i c_i\, \xi_{(i)}(\theta, \phi) = 0$, for all (θ, ϕ) with $c_i = \text{const}$, implies $c_i = 0$.)

(14) There are constants C_{ij}^k, for the three vector fields $\xi_{(i)}$ found in (Ex.13), so that

$$\mathcal{L}_{\xi_{(i)}} \xi_{(j)} = \sum_k C_{ij}^k\, \xi_{(k)}\,. \tag{0.12}$$

Compute these constants.

(15) A geodesic vector field v^i is one which satisfies $v^j \nabla_j\, v^i = 0$. Show that $v = \frac{\partial}{\partial\theta}$ is geodesic for the metric in Ex.(13). Show furthermore, that the curves $x^i(\lambda) = (\theta(\lambda), \phi(\lambda)) = (\lambda, \phi_0)$ are geodesics for all values $\phi_0 \in [0, 2\pi]$.

(16) Conclude from the last statement in Ex.(15) that the vector field $\xi = \frac{\partial}{\partial\phi}$ satisfies the Jacobi equation along the integral curves of v. Use this fact to compute the Riemann tensor of ds^2. Result: $R_{\phi\theta\phi\theta} = \sin^2\theta \implies R = 2$.

(17) A metric g_{ij} is called conformally flat, if there exists a function $F > 0$, so that $g_{ij} = F^2 \mathring{g}_{ij}$, where $\mathring{g}_{ij}$ is a flat metric. Show that ds^2 of Ex.(13) is conformally flat. Hint: Use the transformation $x^1 = \frac{2\sin\theta}{1+\cos\theta}\cos\phi$, $x^2 = \frac{2\sin\theta}{1+\cos\theta}\sin\phi$. Interpret this transformation in terms of stereographic projection from the south pole $\theta = \pi$.

(18) Consider a function F, such that $(\nabla F, \nabla F)$ is constant. Show that the integral curves of $F^\mu := g^{\mu\nu}\nabla_\nu F$ are then geodesics. Show also that this exercise generalizes Ex.(15).

(19) Consider a vector field ξ of the form $\xi = \xi^\mu \partial_\mu = \partial_t$ in some coordinate system (t, x^i). Show that this has all transformations of the form $(t' = t + F(x), x'^i = f^i(x))$ as symmetries.

(20) Consider the 2-dimensional line element given by

$$ds^2 = A(r)\,dt^2 + 2B(r)\,dt\,dr + C(r)\,dr^2 \tag{0.13}$$

Show that, by a suitable transformation, we can arrange for the metric to be diagonal. Hint: Use a transformation under which $\xi = \partial_t$ is invariant.

(21) Consider the radial ($l = 0$) timelike geodesic $r(s)$ in Schwarzschild which starts at $r(0) = R$ with $\dot{r}(0) = 0$. Verify that $(r(z), s(z))$ with

$$r = R\cos^2\frac{z}{2} \qquad\qquad s = \frac{1}{2}\left(\frac{R^3}{2M}\right)^{\frac{1}{2}}(z + \sin z)\ , \tag{0.14}$$

where $z \in [0, \pi)$, gives a parameter representation of the solution. What happens at $r = 2M$? How long does it take the particle to reach $r = 0$?

(22) Let $x^\mu(\lambda)$ be a curve with tangent $v^\mu = \frac{dx^\mu}{d\lambda}$ satisfying

$$v^\nu \nabla_\nu v^\mu = a\, v^\mu \tag{0.15}$$

for some function $a(\lambda)$. Show that, under a change of parametrization $\lambda \mapsto \bar{\lambda} = F(\lambda)$, the form of Eq.(0.15) remains intact, but the function a changes in some way. Then infer that there is always a parametrization so that $\bar{a}$ is zero, and this parametrization is unique up to affine transformations of the form $\bar{\lambda} = A\lambda + B$. Finally show that, for solutions of (0.15) with v^μ timelike, the transition from λ to proper time s also has the effect of rendering a equal to zero.

(23) Using what is largely a repetition of a calculation in the lecture, show that every Killing vector satisfies the identity

$$\nabla_\mu \nabla_\nu \xi_\rho = -R_{\nu\rho\mu}{}^\sigma \xi_\sigma \tag{0.16}$$

(24) Let $h_{\mu\nu}$ and $q_{\mu\nu}$ be respectively the (t, r) - part and (Θ, ϕ) - part of the Schwarzschild metric $g_{\mu\nu}$ (so that $g_{\mu\nu} = h_{\mu\nu} + q_{\mu\nu}$). Then, by a long

calculation, one finds for the Riemann tensor that

$$R_{\mu\nu\rho\sigma} = \frac{4M}{r^3}\left(q_{\rho[\mu}q_{\nu]\sigma} + h_{\rho[\mu}h_{\nu]\sigma} - \frac{1}{2}\,h_{\rho[\mu}q_{\nu]\sigma} + \frac{1}{2}\,h_{\sigma[\mu}q_{\nu]\rho}\right) \tag{0.17}$$

Check that $R_{\mu\nu\rho\sigma}$ has all the necessary symmetries of a Riemann tensor and that $g_{\mu\nu}$ is a vacuum solution. Check furthermore that $r = 0$ is a singularity of some scalar built from the curvature.

(25) Given the additional information that

$$\nabla_\rho\, q_{\mu\nu} = -\frac{2}{r}\, q_{\rho(\mu}\nabla_{\nu)}r \tag{0.18}$$

show that this Riemann tensor also satisfies the (differential) Bianchi identities $\nabla_{[\lambda}R_{\mu\nu]\rho\sigma} = 0$. Hint: Use $h_{\rho[\mu}r_\lambda h_{\nu]\rho} = 0$, where $r_\lambda = \nabla_\lambda r$.

(26) Show that the Schwarzschild coordinate r extends to a globally defined function on the Kruskal manifold and that it has saddle points on the 'bifurcation 2-sphere', where the future ('$t = \infty$') and past ('$t = -\infty$') horizons intersect.

(27) A 'relativistic spacetime with c' consists of a family of Lorentz metrics $g^c_{\mu\nu}$ of signature $(+,-,-,-)$ called 'time metrics', such that both $g^c_{\mu\nu}$ and the (inverse) 'space metrics' $h^{\mu\nu}_c$ given by $h^{\mu\nu}_c g^c_{\nu\rho} = -\frac{1}{c^2}\,\delta^\mu{}_\rho$ have limits as $c \to \infty$. These limit metrics $g^\infty_{\mu\nu}$ and $h^{\mu\nu}_\infty$ are of course degenerate. Construct such sequences for the Minkowski metric and compute the signature of $g^\infty_{\mu\nu}$ and $h^{\mu\nu}_\infty$. Then reconsider the nonrelativistic limit of $\nabla_\nu T^{\mu\nu} = 0$ in the lecture with

$$T^{\mu\nu}_c = \left(\rho + \frac{p}{c^2}\right)\, u^\mu_c u^\nu_c + p\, h^{\mu\nu}_c \tag{0.19}$$

and where u^μ_c is normalized by $g^c_{\mu\nu}u^\mu_c u^\nu_c = 1$.

(28) From the divergence-free-condition for the energy momentum tensor of a perfect fluid, i.e.

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p\, g^{\mu\nu} \tag{0.20}$$

conclude that

$$\nabla_\mu(\rho u^\mu) + p\nabla_\mu u^\mu = 0 \tag{0.21}$$

Hint: contract $\nabla_\nu T^{\mu\nu}$ with u_μ and that $u_\mu\nabla_\nu u^\mu$ is zero. Eq.(0.21) is the relativistic version of the continuity equation in the presence of pressure.

(29) When $\nabla_\nu T^{\mu\nu} = 0$ and ξ^μ is a Killing vector, show that there holds the conservation law

$$\nabla_\nu(T^\nu{}_\mu\xi^\mu) = 0 \tag{0.22}$$

Suppose we are in Minkowski space. Make sense of the following statement: 'The symmetry $T_{\mu\nu} = T_{\nu\mu}$ of the energy momentum tensor is relevant for angular momentum conservation, but not for linear momentum conservation.' Hint: For space and time translation Killing vectors there holds that $\nabla_\mu \xi^\nu = 0$, which is stronger than the Killing equation.

(30) In the case of a scalar field Φ obeying $\Box\Phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi = 0$, show that $T^{\mu\nu}$ defined by

$$T_{\mu\nu} = (\nabla_\mu\Phi)(\nabla_\nu\Phi) - \frac{1}{2}\, g_{\mu\nu}(\nabla\Phi)^2 \tag{0.23}$$

satisfies $\nabla_\nu T^{\mu\nu} = 0$. When $(M, g_{\mu\nu})$ is Minkowski space and ξ^μ the Killing vector ∂_t, show that the associated conservation law leads to the conservation of total energy demonstrated in the lecture.

(31) Compute the area of a surface of constant isotropic radius ρ in a $t =$ const -slice of the Schwarzschild metric and show it has a minimum at $\rho = \frac{m}{2}$.

(32) A timelike Killing vector ξ^μ is called static, when $\xi_{[\mu}\nabla_\nu\xi_{\rho]} = 0$. Prove that it then satisfies $\nabla_\mu\xi_\nu = -\frac{1}{\xi^2}\,\xi_{[\mu}\nabla_{\nu]}\xi^2$, where $\xi^2 = g_{\mu\nu}\xi^\mu\xi^\nu$. Hint: Use the general fact that $a_{[\mu}b_{\nu\rho]} = 0$, with a_μ nowhere zero and $b_{\mu\nu} = b_{[\mu\nu]}$, implies that there exists c_μ, such that $b_{\mu\nu} = a_{[\mu}c_{\nu]}$. Alternatively you might start out by expanding $\xi_{[\mu}\nabla_\nu\xi_{\rho]} = 0$ and then contracting with ξ^μ.

(33) When

$$\Box\gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\,\Box\gamma - 2\gamma^\rho{}_{(\mu,\nu)\rho} = 0 \tag{0.24}$$

where $\gamma_{\mu\nu} = \mathrm{Re}(c_{\mu\nu}\, e^{i(k,x)})$ with $c_{\mu\nu} = \text{const}$, $k_\mu = \text{const}$ and $(k,k) = 0$, show that $\gamma^\mu{}_{\nu,\mu} = 0$ holds automatically.

(34) Let $T^{\mu\nu}(t,x)$ be a symmetric and divergence-free tensor on Minkowski space which is zero for large $|x|$. Prove the 'Laue-theorem'

$$\int_{\mathbb{R}^3} T_{ij}(t,x)\, d^3x = \frac{1}{2}\frac{d^2}{dt^2}\int_{\mathbb{R}^3} T_{00}(t,x)\, x_i x_j\, d^3x \tag{0.25}$$

(35) Consider the vector field ξ on Minkowski space given by $\xi^\mu\partial_\mu = \partial_t + \Omega\partial_\phi$ with $\Omega = \text{const}$. Discuss the causal nature of ξ in the different regions of Minkowski space.

(36) Is ξ in the previous exercise hypersurface-orthogonal, i.e. is the quantity $\xi_{[\mu}\nabla_\nu\xi_{\rho]}$ is zero?

(37) Show that a curve tangent to a Killing vector is a geodesic if the norm of the Killing vector has zero gradient along this curve.

(38) Apply the previous result to find the timelike, spatially circular geodesics of the Schwarzschild spacetime by considering $\xi^\mu\partial_\mu = \partial_t + \Omega\,\partial_\phi$ (in Schwarzschild coordinates). Answer: $r^3 = \frac{M}{\Omega^2}\,, \theta = \frac{\pi}{2}$. Show also that

$r > 3M$.

(39) Prove that

$$(\rho R^3)^{\cdot} + p(R^3)^{\cdot} = 0 \tag{0.26}$$

for the standard cosmological model.

(40) Show (0.26) directly from the contracted Bianchi identities.

(41) Show that the metric

$$g_{ij}\, dx^i dx^j = \frac{dr^2}{1-r^2} + r^2 d\Omega^2 \tag{0.27}$$

can be realized as the metric on $\mathbb{S}^3$, i.e. the metric induced by the Euclidean metric $\delta_{\mu\nu}$ on $\mathbb{R}^4$ on the three surface given by $\delta_{\mu\nu} x^\mu x^\nu = 1$.

(42) Compute the volume of $(\mathbb{S}^3, g_{ij})$.

(43) Show that the metric

$$g_{ij}\, dx^i dx^j = \frac{dr^2}{1+r^2} + r^2 d\Omega^2 \tag{0.28}$$

can be realized as the metric on $\mathbb{H}^3$, i.e. the metric induced by the Minkowski metric on $\mathbb{R}^4$ on the three surface given by $\eta_{\mu\nu} x^\mu x^\nu = -1$.

(44) Prove for the Schwarzschild metric that (notation as in Ex.(24))

$$\nabla_\mu \nabla_\nu r = \frac{1}{r}\left[\frac{M}{r} g_{\mu\nu} + \left(1 - \frac{3M}{r}\right) q_{\mu\nu}\right] \tag{0.29}$$

(45) Suppose two metrics $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ are related by

$$\bar{g}_{\mu\nu} = \omega^2 g_{\mu\nu}\,, \quad \omega > 0 \tag{0.30}$$

Show that the corresponding Christoffel symbols are related by

$$\bar{\Gamma}^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} + \omega^{-1}(2\,\delta^\mu{}_{(\nu}\nabla_{\rho)}\omega - g^{\mu\sigma} g_{\nu\rho} \nabla_\sigma \omega) \tag{0.31}$$

Use this result to show that the concept of null geodesics is conformally invariant (whereas that of affine parameter is not).

(46) De Sitter spacetime is given by the metric ($S > 0$, $0 < \chi < \pi$)

$$g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + S^2 \cosh^2(S^{-1} t)(d\chi^2 + \sin^2 \chi\, d\Omega^2) \tag{0.32}$$

on $\mathbb{R} \times \mathbb{S}^3$. Argue that this spacetime satisfies

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{0.33}$$

with $S^{-1} = \sqrt{\frac{\Lambda}{3}}$.

(47) Realize the $(1+1)$ - version of (0.32), i.e.

$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + S^2\cosh^2(S^{-1}t)\,d\phi^2 \tag{0.34}$$

as a submanifold of 3-dimensional Minkowski space.

(48) A vector field ξ is called a conformal Killing vector, when $\mathcal{L}_\xi g_{\mu\nu} = \Phi g_{\mu\nu}$ for some function Φ. Show that $\xi = R(t)\,\partial_t$ is a conformal Killing vector on Robertson Walker spacetimes.

(49) A vector field $\xi^\mu(x)$ is tangent to a 2-surface Σ given by $x^\mu = f^\mu(y^A)$ $(A = 1, 2)$, when there exists $X^A(y)$ such that $\xi^\mu(f(y)) = f^\mu{}_{,A}(y)X^A(y)$. Let ξ and η be vector fields tangent to Σ. Show that their Lie bracket is also tangent to Σ.

(50) Let ξ^μ be a timelike Killing vector satisfying $\omega_{\mu\nu\rho} := \xi_{[\mu}\nabla_\nu\xi_{\rho]} = 0$ and (t, x^i) coordinates such that $\xi^\mu\partial_\mu = \partial_t$. Show that there exists a change of coordinates $\bar{t} = t - F(x^i)$, $\bar{x}^i = x^i$, so that $\bar{g}_{0i} = 0$.

(51) Let $\sigma_{\mu\nu}$ be antisymmetric. Express the orthogonal projection of $\sigma_{\mu\nu}$ with respect to some timelike vector ξ^μ in terms of the contraction of $\xi_{[\mu}\sigma_{\nu\rho]}$ with ξ^μ.

(52) Let ξ^μ be a timelike Killing field and S^μ a vector orthogonal to ξ^μ, which is Fermi transported along a trajectory of ξ^μ. Prove that

$$\mathcal{L}_\xi S_\mu = \frac{3}{\xi^2}\,\omega_{\mu\nu\rho}\,S^\nu\xi^\rho\,, \tag{0.35}$$

where $\xi^2 = \xi^\mu\xi_\mu$.

(53) Compare the proper time τ along a period for a circular geodesic of radius R in Schwarzschild with the proper time s for a static observer at the same radius. Show that

$$\frac{s}{\tau} = \sqrt{\frac{1 - \frac{2M}{R}}{1 - \frac{3M}{R}}} \tag{0.36}$$