General Relativity I, C7.2b, Hilary Term 2009

Piotr T. Chruściel
Mathematical Institute and Hertford College
University of Oxford

March 2010

Contents

Chapter 1

Introductory material

1.1 Synopsis (from the course web page, from URL `https://www.maths.ox.ac.uk/node/4068`

1.1.1 Recommended Prerequisites

B7.2a Relativity and Electromagnetism.

1.1.2 Aims & Objectives

The course is intended as an elementary introduction to general relativity, its basic physical concepts of its observational implications, and the new insights that it provides into the nature of space time, and the structure of the universe. Familiarity with special relativity and electromagnetism as covered in the B7 course will be assumed. The lectures will review Newtonian gravitation, tensor calculus and continuum physics in special relativity, physics in curved space time and the Einstein field equations. This will suffice for an account of simple applications to planetary motion, the bending of light and the existence of black holes.

This course starts by asking how the theory of gravitation can be made consistent with the special-relativistic framework. Physical considerations (the principle of equivalence, general covariance) are used to motivate and illustrate the mathematical machinery of tensor calculus. The technical development is kept as elementary as possible, emphasising the use of local inertial frames. A similar elementary motivation is given for Einstein's equations and the Schwarzschild solution. Orbits in the Schwarzschild solution are given a unified treatment which allows a simple account of the three classical tests of Einstein's theory. Finally, the analysis of extensions of the Schwarzschild solution show how the theory of black holes emerges and exposes the radical consequences of Einstein's theory for space-time structure. Cosmological solutions are not discussed.

1.1.3 Learning Outcomes

Students will have developed a knowledge and appreciation of the ideas and concepts described above.

1.1.4 Synopsis

Review of Newtonian gravitation theory and problems of constructing a relativistic generalisation. Review of Special Relativity. The equivalence principle. Tensor formulation of special relativity (including general particle motion, tensor form of Maxwell's equations and the energy momentum-tensor of dust). Curved space time. Local inertial coordinates. General coordinate transformations, elements of Riemannian geometry (including connections, curvature and geodesic deviation). Mathematical formulation of General Relativity, Einstein's equations (properties of the energy-momentum tensor will be needed in the case of dust only). The Schwarzschild solution; planetary motion, the bending of light, and black holes.

1.1.5 Reading List

1. L.P. Hughston and K.P. Tod, An Introduction to General Relativity, LMS Student Text 5, CUP (1990), Chapters 1-18.

2. N.M.J. Woodhouse, Notes on Special Relativity, (Mathematical Institute Notes. Revised edition; published in a revised form as Special Relativity, Lecture notes in Physics m6, Springer Verlag, (1992), Chs 1-7

Further Reading

1. B. Schutz, A First Course in General Relativity, CUP (1990).
2. R.M. Wald, General Relativity, Chicago (1984).
3. W. Rindler, Essential Relativity, Springer Verlag, 2nd edition (1990).

1.2 Lectures 1-2

see Woodhouse [9], Chapters 1-2

You are expected to study, and master, both what has been covered in the lectures, and that relevant part of the material in Woodhouse's book which has not.

See also the excellent entry in Wikipedia on the Principle of Equivalence, URL `http://en.wikipedia.org/wiki/Principle_of_equivalence`

1.3 Warning

Whatever follows are very rough notes, hardly proofread. All civilised and constructive comments about typos, mistakes, etc., are welcome.

Chapter 2

Introduction to tensor calculus

2.1 Some reminders: Minkowski space-time

A *quadratic form* is a homogeneous quadratic polynomial on a vector space, say V. So, in finite dimension n, if $X \in V$ is decomposed as $X^i e_i$ in a basis e_i, $i = 1, \ldots, n$, then a quadratic form Q can be uniquely represented as

$$Q(X) = \sum_{i=1}^{n} Q_{ij} X^i X^j \ ,$$

for a collection of numbers Q_{ij}. Note that the antisymmetric part of Q_{ij}, if any, would give zero contribution to the sum, and therefore without loss of generality we can assume that Q_{ij} is symmetric:

$$Q_{ij} = Q_{ji} \ .$$

It is known that every quadratic form can be written as a sum of squares, with coefficients ± 1. We say that the signature is (p, q) if there exists a representation with p pluses and q minuses. The signature is sometimes also written as

$$(\underbrace{+, \ldots, +}_{p \text{ times}}, \underbrace{-, \ldots, -}_{q \text{ times}}) \ .$$

The $(n+1)$-dimensional Minkowski space-time is the vector space $\mathbb{R}^{n+1}$ equipped with a quadratic form η with signature $(1, n)$, equivalently $(+, -, \ldots, -)$. Points $x \in \mathbb{R}^{n+1}$ will often be written as $(t, \vec{x})$.

By the *Sylvester inertia theorem* the signature is coordinate-independent, and (as already pointed out), there exists a choice of coordinates x^μ, called *inertial coordinates*, so that •2.1.1

$$\eta(x, y) = x^0 y^0 - x^1 y^1 - \ldots - x^n y^n \ . \tag{2.1.1}$$

•2.1.1: This is the familiar special relativistic "line element", using units in which the speed of light c equals 1!!!

or

$$\eta(x,x) = x^0x^0 - x^1x^1 - \ldots - x^nx^n \ . \tag{2.1.2}$$

(Here he have implicitly used the obvious one-to-one correspondence between quadratic forms, such as the right-hand-side of (2.1.2), with *bilinear forms*, such as the right-hand-side of (2.1.1)). One also writes $\eta = (\eta_{\mu\nu})$, representing η by a $(n+1)\times(n+1)$ matrix

$$(\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \ldots & 0 & -1 \end{pmatrix} ,$$

so

$$\eta(x,y) = \eta_{\mu\nu}x^\mu y^\nu \ .$$

Here we are using the summation convention, which means that repeated indices, one in subscript and one in superscript position, have to be summed over. For example, if greek indices are assumed to run from 0 to n, then (2.1.3) below means

$$x^\mu = L^\mu{}_0\tilde{x}^0 + L^\mu{}_1\tilde{x}^1 + \ldots + L^\mu{}_n\tilde{x}^n \ .$$

A vector X is called *timelike* if $\eta(X,X) > 0$, null if $X \neq 0$ and $\eta(X,X) = 0$, spacelike if $\eta(X,X) < 0$. An example of a null vector is given by the vector with components $(1,1,0,0)$ (in a coordinate system in which η takes the form (2.1.2)).

We write $\mathbb{R}^{1,n}$ for

$$\mathbb{R}^{1,n} = (\mathbb{R}^{n+1}, \eta) \ .$$

The coordinates in which η takes the form (2.1.1) are called *inertial*, or *non-accelerating*. Two such coordinate systems x^μ and $\tilde{x}^\mu$ differ by a Lorentz transformation:

$$x^\mu = L^\mu{}_\nu\tilde{x}^\nu \ , \tag{2.1.3}$$

where $L = (L^\mu{}_\nu)$ is a Lorentz matrix.

A useful result about inertial coordinates is the following: given any timelike vector $T = (T^\mu)$, there exists an inertial frame in which the T has components $(t,0,\ldots,0)$, for some $t \in \mathbb{R}^*$. (There are in fact many such frames, differening from each other by a rotation, possibly composed with a parity transformation and/or a time reversal.) This is proved by a Gram-Schmidt orthogonalisation, using a basis in which T is the first basis vector.

By definition, Lorentz matrices are those matrices which preserve the quadratic form η; in matrix notation:

$$\eta_{\mu\nu} = \eta_{\alpha\beta}L^\alpha{}_\mu L^\beta{}_\nu = \underbrace{L^\alpha{}_\mu\eta_{\alpha\beta}L^\beta{}_\nu}_{L^t\eta L} ,$$

This implies $\det L = \pm 1$.

Examples are:

1. a boost along the x-axis

$$L = (L^\mu{}_\nu) = \begin{pmatrix} \gamma & \gamma u & 0 & 0 \\ \gamma u & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \qquad |u| < 1 \, , \quad \gamma = \frac{1}{1-u^2}$$

2. rotations in space: $t = \tilde{t}$, $\vec{x} = R\vec{\tilde{x}}$, R – rotation matrix

3. time reversal: $t = -\tilde{t}$, $\vec{x} = \vec{\tilde{x}}$,

4. space inversion: $t = \tilde{t}$, $\vec{x} = -\vec{\tilde{x}}$,

THEOREM 2.1.1 *Every Lorentz transformation is a composition of a number of the transformations above.*

DEFINITION 2.1.2 *Poincaré transformations are compositions of Lorentz transformations and translations.*

•2.1.2 We shall soon be using the symbol η for the following object:

$$\eta = dt^2 - (dx^1)^2 - \ldots - (dx^n)^2 \, , \tag{2.1.4}$$

•*2.1.2: smaller font paragraphs, such as this one, can (and should) be safely ignored during a first reading, and are not examinable*

which means something closely related to (2.1.2), but not quite the same thing. In (2.1.2) η is a quadratic form on $\mathbb{R}^{n+1}$, while η in (2.1.4) is a *field of quadratic forms* on $\mathbb{R}^{n+1}$: this means that at each point $x \in \mathbb{R}^{n+1}$, η is a quadratic form on the "tangent space $T_x\mathbb{R}^{n+1}$": this is the collection of all vectors tangent to $\mathbb{R}^{+1}$ at the point x. This last space is linearly isomorphic to $\mathbb{R}^{n+1}$, with the "tangent bundle" $T\mathbb{R}^{n+1} := \cup_{x\in\mathbb{R}^{n+1}} T_x\mathbb{R}^{n+1}$ being diffeomorphic to $\mathbb{R}^{2(n+1)}$; we will return to this shortly, see Section 2.2.2.

2.1.1 Electromagnetic fields

In everyday experience, the electromagnetic field is thought of as two vector fields, $\vec{E}$ and $\vec{B}$. If there are no sources, $\vec{E}$ and $\vec{B}$ satisfy the following sourceless Maxwell equations:

$$\nabla \cdot \vec{E} = 0 \, , \qquad \nabla \cdot \vec{B} = 0 \, , \tag{2.1.5}$$

$$\underbrace{\mu_0 \epsilon_0}_{=1} \frac{\partial \vec{E}}{\partial t} = \nabla \wedge \vec{B} \, , \qquad \frac{\partial \vec{B}}{\partial t} = -\nabla \wedge \vec{E} \, , \tag{2.1.6}$$

Lorentz discovered his group by studying the invariance properties of those equations under linear coordinate transformations.

The bottom-line of his calculation is, that if you form an anti-symmetric matrix F, called *the Maxwell tensor*, as follows:

$$F = (F^{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} ,$$

then F transforms as

$$F = L\tilde{F}L^t \quad \Longleftrightarrow \quad F^{\mu\nu} = L^\mu{}_\alpha \tilde{F}^{\alpha\beta} L^\nu{}_\beta$$

In fact, Lorentz proved:

THEOREM 2.1.3 *The Maxwell equations are preserved by linear transformations of coordinates and of $\vec{E}$ and $\vec{B}$ if and only if the above transformation laws hold, where Λ is the matrix of a Lorentz transformation.*

Let ϵ_{ijk} be the three-dimensional alternating symbol, this means that: $\epsilon_{123} = 1$, and $\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{jki}$; you should check that these equations define all components of ϵ_{ijk} uniquely. Then we have (please check as well!)

$$F_{ij} = \epsilon_{ijk} B^k ,$$

keeping in mind that $F^{\mu\nu}$ is obtained from $F_{\mu\nu}$ by raising indices:

$$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta} F_{\alpha\beta} ,$$

and that, for consistency, $B_i = -B^i$, $E_i = -E^i$.

EXERCICE 2.1.4 Relate the set of equations

$$\partial_\mu F^{\mu\nu} = 0$$

to the Maxwell equations (2.1.5)-(2.1.6).

2.2 Introduction to tensor calculus

We have seen so far at least two objects with rather different behavior under changes of inertial frames — the frames themselves, and the Maxwell tensor F. It makes sense to try to systematise all this. •2.2.1

•2.2.1: *In addition to the current notes on the introduction to tensor calculus, you should study the relevant parts of Woodhouse's book.*

2.2.1 Scalar functions

Under a change of coordinates $x^i \to y^j(x^i)$ a scalar function f simply changes using composition; so to a function $f(x)$ we associate a new function

$$\bar{f}(y) = f(x(y)) \, .$$

In general relativity it is a common abuse of notation to write the same symbol f for what we wrote $\bar{f}$, when we think that this is the same function but expressed in a different coordinate system. In this section, to make things clearer, we will make this notational distinction, but this will almost never be done in the remainder of the lectures. For example we will systematically use the same symbol $g_{\mu\nu}$ for the metric components, whatever the coordinate system used.

2.2.2 Vector fields

The notion of a vector field finds its roots in the notion of *tangents to a curve*, say $s \to \gamma(s)$.

If we use local coordinates to write $\gamma(s)$ as $(\gamma^1(s), \gamma^2(s), \ldots, \gamma^n(s))$, the tangent to that curve at the point $\gamma(s)$ is defined as the set of numbers

$$(\dot{\gamma}^1(s), \dot{\gamma}^2(s), \ldots, \dot{\gamma}^n(s)) \, ,$$

where a dot denotes a derivative with respect to s.

Consider a curve $\gamma(s)$ given in a coordinate system x^i and let us perform a change of coordinates $x^i \to y^j(x^i)$. In the new coordinates y^j the curve γ is represented by the functions $y^j(\gamma^i(s))$, with new tangent

$$\frac{dy^j}{ds} = \sum_i \frac{\partial y^j}{\partial x^i} \dot{\gamma}^i$$

This formula defines what is called *the transformation law of vectors*: given a point $x = (x^i)$ and a set of numbers $X = (X^i)$, the set (X^i) is called a *vector at* x if, under a change of coordinates $x^i \to y^j(x^i)$ the set (X^i) transforms as

$$X^i(x) \to \bar{X}^j(y(x)) = \sum_i \frac{\partial y^j}{\partial x^i}(x) X^i(x) \, , \tag{2.2.1}$$

so that finally

$$\bar{X}^j(y) = \sum_i \frac{\partial y^j}{\partial x^i}(x(y)) X^i(x(y)) \, , \tag{2.2.2}$$

A very convenient way of representing vectors is using *first order homogeneous differential operators*. So, consider a vector field represented in a coordinate system x^i by a set of functions (X^i), which transform using the transformation rule (2.2.1) under coordinate changes. Define the following differential operator

$$X := X^1 \frac{\partial}{\partial x^1} + \ldots + X^n \frac{\partial}{\partial x^n} \ .$$

The point is that *the transformation rule (2.2.1) becomes implicit in the notation.* Indeed, consider a function f, so that the differential operator X acts on f by differentiation:

$$X(f)(x) := \sum_i X^i \frac{\partial f(x)}{\partial x^i} \ . \tag{2.2.3}$$

If we make a coordinate change so that

$$x^j = x^j(y^k) \quad \Longleftrightarrow \quad y^k = y^k(x^j) \ ,$$

keeping in mind that

$$\bar{f}(y) = f(x(y)) \quad \Longleftrightarrow \quad f(x) = \bar{f}(y(x)) \ ,$$

then

$$\begin{aligned} X(f)(x) \ &:= \ \sum_i X^i(x) \frac{\partial f(x)}{\partial x^i} \\ &= \ \sum_i X^i(x) \frac{\partial \bar{f}(y(x))}{\partial x^i} \\ &= \ \sum_{i,k} X^i(x) \frac{\partial \bar{f}(y(x))}{\partial y^k} \frac{\partial y^k}{\partial x^i}(x) \\ &= \ \sum_k \bar{X}^k(y(x)) \frac{\partial \bar{f}(y(x))}{\partial y^k} \\ &= \ \left(\sum_k \bar{X}^k \frac{\partial \bar{f}}{\partial y^k} \right) (y(x)) \ , \end{aligned}$$

with $\bar{X}^k$ given by (2.2.1). So

$\boxed{X(f) \text{ is a scalar iff the coefficients } X^i \text{ satisfy the transformation law of a vector.}}$

Exercice 2.2.1 Check that this is a necessary and sufficient condition.

From now on the summation convention will be used systematically, so that the summation symbol will be omitted whenever in a formula

two identical indices occur, one up and one down, unless explicitly stated otherwise.

We will often use the middle formula in the above calculation in the form

$$\frac{\partial}{\partial x^i} = \frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial y^k} . \tag{2.2.4}$$

Note that the tangent to the curve $s \to (s, x^2, x^3, \ldots x^n)$, where $(x^2, x^3, \ldots x^n)$ are constants, is identified with the differential operator

$$\partial_1 \equiv \frac{\partial}{\partial x^1}$$

Similarly the tangent to the curve $s \to (x^1, s, x^3, \ldots x^n)$, where $(x^1, x^3, \ldots x^n)$ are constants, is identified with

$$\partial_2 \equiv \frac{\partial}{\partial x^2}$$

and then $\dot\gamma$ is identified with

$$\dot\gamma(s) = \dot\gamma^i \partial_i$$

2.2.3 Covectors

Covectors are *linear maps on the space of vectors.*

The basic object is the *coordinate differential* dx^i, defined by its action on vectors as follows:

$$dx^i(X^j\partial_j) := X^i . \tag{2.2.5}$$

Equivalently,

$$dx^i(\partial_j) := \delta^i_j := \left\{ \begin{array}{ll} 1, & i = j; \\ 0, & \text{otherwise.} \end{array} \right.$$

The dx^i's form a basis for the space of covectors: indeed, let φ be a linear map on the space of vectors, then

$$\varphi(\underbrace{X}_{X^i\partial_i}) = \varphi(X^i\partial_i) \underbrace{=}_{\text{linearity}} X^i \underbrace{\varphi(\partial_i)}_{\text{call this } \varphi_i} = \varphi_i dx^i(X) \underbrace{=}_{\text{def. of sum of functions}} (\varphi_i dx^i)(X) ,$$

hence

$$\varphi = \varphi_i dx^i ,$$

and every φ can indeed be written as a linear combination of the dx^i's. Under a change of coordinates we have

$$\bar\varphi_i \bar X^i = \bar\varphi_i \frac{\partial y^i}{\partial x^k} X^k = \varphi_k X^k ,$$

leading to the following transformation law for components of covectors:

$$\varphi_k = \bar{\varphi}_i \frac{\partial y^i}{\partial x^k} , \tag{2.2.6}$$

Given a scalar f, we define its *differential* df as

$$df = \frac{\partial f}{\partial x^1} dx^1 + \ldots + \frac{\partial f}{\partial x^n} dx^n .$$

With this definition, dx^i *is* the differential of the coordinate function x^i.

As presented above, the differential of a function is a covector by definition. As an exercice, you should check directly that the collection of functions $\varphi_i := \partial_i f$ satisfies the transformation rule (2.2.6).

We have a formula which is often used in calculations

$$dy^j = \frac{\partial y^j}{\partial x^k} dx^k .$$

An elegant approach to the definition of differentials proceeds as follows: Given any function f, we define:

$$df(X) := X(f) . \tag{2.2.7}$$

(Recall that here we are viewing a vector field X as a differential operator on functions, defined by (2.2.3).) The map $X \mapsto df(X)$ is linear under addition of vectors, and multiplication of vectors by numbers: if λ, μ are real numbers, and X and Y are vector fields, then

$$\begin{aligned} df(\lambda X + \mu Y) & \underbrace{=}_{\text{by definition (2.2.7)}} (\lambda X + \mu Y)(f) \\ & \underbrace{=}_{\text{by definition (2.2.3)}} \lambda X^i \partial_i f + \mu Y^i \partial_i f \\ & \underbrace{=}_{\text{by definition (2.2.7)}} \lambda df(X) + \mu df(Y) . \end{aligned}$$

Applying (2.2.7) to the function $f = x^i$ we obtain

$$dx^i(\partial_j) = \frac{\partial x^i}{\partial x^j} = \delta^i_j ,$$

recovering (2.2.5).

EXAMPLE 2.2.2 Let (ρ, φ) be polar coordinates on $\mathbb{R}^2$, thus $x = \rho \cos \varphi$, $y = \rho \sin \varphi$, and then

$$\begin{aligned} dx &= d(\rho \cos \varphi) = \cos \varphi d\rho - \rho \sin \varphi d\varphi , \\ dy &= d(\rho \sin \varphi) = \sin \varphi d\rho + \rho \cos \varphi d\varphi . \end{aligned}$$

At any given point p of a manifold M the set of vectors forms a vector space, denoted by T_pM. The collection of all the tangent spaces is called the tangent bundle to M, denoted by TM.

Similarly, at any given point $p \in M$ the set of covectors forms a vector space, denoted by T_p^*M. The collection of all the tangent spaces is called the cotangent bundle to M, denoted by T^*M.

2.2.4 Bilinear maps, two-covariant tensors

A map is said to be multi-linear if it is linear in every entry; e.g. g is bilinear if

$$g(aX + bY, Z) = ag(X, Z) + bg(Y, Z) ,$$

and

$$g(X, aZ + bW) = ag(X, Z) + bg(X, W) .$$

A map g which is bilinear on the space of vectors can be represented by a matrix with two indices down:

$$g(X, Y) = g(X^i\partial_i, Y^j\partial_j) = X^iY^j \underbrace{g(\partial_i, \partial_j)}_{=:g_{ij}} = g_{ij}dx^i(X)dx^j(Y)$$

We say that g is a *covariant tensor of valence two*.

We say that g is *symmetric* if $g(X, Y) = g(Y, X)$ for all X, Y; equivalently, $g_{ij} = g_{ji}$.

A symmetric bilinear tensor field is said to be *non-degenerate* if $\det g_{ij}$ has no zeros.

By Sylvester's inertia theorem, there exists a basis θ^i of the space of covectors so that a symmetric bilinear map g can be written as

$$g(X, Y) = \theta^1(X)\theta^1(Y) + \ldots + \theta^s(X)\theta^s(Y) - \theta^{s+1}(X)\theta^{s+1}(Y) - \ldots - \theta^{s+r}(X)\theta^{s+r}(Y)$$

(s, r) is called the signature of g; in geometry, unless specifically said otherwise, one always assumed that the signature does not change from point to point.

If $s = n$, in dimension n, then g is said to be a Riemannian metric tensor.

A canonical example is provided by the flat Riemannian metric on $\mathbb{R}^n$ is

$$g = (dx^1)^2 + \ldots + (dx^n)^2 .$$

By definition, a *Riemannian metric* is a field of symmetric two-covariant tensors with signature $(+, \ldots, +)$ and with $\det g_{ij}$ without zeros.

A Riemannian metric can be used to define the length of curves: if $\gamma : [a,b] \ni s \to \gamma(s)$, then

$$\ell_g(\gamma) = \int_a^b \sqrt{g(\dot\gamma,\dot\gamma)}ds \ .$$

One can then define the distance between points by minimizing the length of the curves connecting them.

If $s = 1$ and $r = N-1$, in dimension N, then g is said to be a *Lorentzian metric tensor*.

For example, the *Minkowski metric* on $\mathbb{R}^{1+n}$ is

$$\eta = (dx^0)^2 - (dx^1)^2 - \ldots - (dx^n)^2 \ .$$

2.2.5 Tensor products

If φ and θ are covectors, we can define a bilinear map using the formula

$$(\varphi \otimes \theta)(X,Y) = \varphi(X)\theta(Y) \ . \tag{2.2.8}$$

For example

$$(dx^1 \otimes dx^2)(X,Y) = X^1Y^2 \ .$$

Using this notation, if g is a bilinear map on vectors, then we have

$$g(X,Y) = g(X^i\partial_i, Y^j\partial_j) = \underbrace{g(\partial_i,\partial_j)}_{=:g_{ij}} \underbrace{\underbrace{X^i}_{dx^i(X)} \underbrace{Y^j}_{dx^j(Y)}}_{(dx^i\otimes dx^j)(X,Y)} = (g_{ij}dx^i \otimes dx^j)(X,Y) \ . \tag{2.2.9}$$

We will write dx^idx^j for the symmetric product,

$$dx^idx^j := \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i) \ ,$$

so that (2.2.9) is most of the time written as

$$g(X,Y) = (g_{ij}dx^idx^j)(X,Y) \quad \Longleftrightarrow \quad g = g_{ij}dx^idx^j \ .$$

This formula allows one to read-off, without even having to think, the transformation law of a metric tensor under coordinate changes:

$$g_{ij}(x) \underbrace{dx^i}_{\frac{\partial x^i}{\partial y^k}dy^k} \underbrace{dx^j}_{\frac{\partial x^j}{\partial y^\ell}dy^\ell} = g_{ij}(x)\frac{\partial x^i}{\partial y^k}\frac{\partial x^j}{\partial y^\ell}dy^kdy^\ell \ .$$

So, if we write $\bar g_{k\ell}(y)dy^kdy^\ell$ for the metric in the coordinates y, we have

$$\bar g_{kl}(y) = g_{ij}(x(y))\frac{\partial x^i}{\partial y^k}(y)\frac{\partial x^j}{\partial y^\ell}(y) \ .$$

EXAMPLE 2.2.3 Let (ρ, φ) be polar coordinates on $\mathbb{R}^2$:

$$x = \rho \cos\varphi \ , \quad y = \rho \sin\varphi \ .$$

We then have

$$dx = d(\rho\cos\varphi) = \cos\varphi\, d\rho - \rho\sin\varphi\, d\varphi \ , \quad y = d(\rho\sin\varphi) = \sin\varphi\, d\rho + \rho\cos\varphi\, d\varphi \ .$$

From this, we find that the Euclidean metric $g = dx^2 + dy^2$ on the plane can also be written as

$$\begin{aligned} g &= dx^2 + dy^2 = (\cos\varphi\, d\rho - \rho\sin\varphi\, d\varphi)^2 + (\sin\varphi\, d\rho + \rho\cos\varphi\, d\varphi)^2 \\ &= d\rho^2 + \rho^2 \sin^2\varphi\, d\varphi^2 \ , \end{aligned}$$

Recall that, by definition, a Lorentzian metric is a symmetric two-covariant tensor field with signature $(+, -, \ldots, -)$. This implies that the determinant $\det g_{\mu\nu}$ of the matrix $g_{\mu\nu}$ representing g in some coordinate system is strictly negative; given a symmetric bilinear tensor, calculating this determinant provides a good hint whether or not this might be a Lorentzian metric.

One writes $dx^i \wedge dx^j$ for the anti-symmetric tensor product:

$$dx^i \wedge dx^j := \frac{1}{2}(dx^i \otimes dx^j - dx^j \otimes dx^i) \ .$$

It should be clear how this generalises: the tensors $dx^i \otimes dx^j \otimes dx^k$, defined as

$$(dx^i \otimes dx^j \otimes dx^k)(X, Y, Z) = X^i Y^j Z^k \ ,$$

form a basis for three-linear maps on the space of vectors. In other words, in local coordinates, every three-linear map X can be written in the form

$$X = X_{ijk} dx^i \otimes dx^j \otimes dx^k \ .$$

Here X is a tensor of valence $(0, 3)$. Each index leads to a transformation factor as in a covector:

$$X = X_{ijk} dx^i \otimes dx^j \otimes dx^k = X_{ijk} \frac{\partial x^i}{\partial y^m} \frac{\partial x^j}{\partial y^\ell} \frac{\partial x^k}{\partial y^n} dy^m \otimes dy^\ell \otimes dy^n \ .$$

It is sometimes useful to think of vectors as linear maps on co-vectors, using a formula which looks funny when first met: if θ is a covector, and X is a vector, then

$$X(\theta) := \theta(X)$$

(the right-hand-side is the value of θ on X, so this formula defines the left-hand-side). So if $\theta = \theta_i dx^i$ and $X = X^i \partial_i$ then

$$\theta(X) = \theta_i X^i = X^i \theta_i = X(\theta) \ .$$

It then makes sense to define e.g. $\partial_i \otimes \partial_j$ as a bilinear map on covectors: if θ and ψ are covectors, then

$$(\partial_i \otimes \partial_j)(\theta, \psi) := \theta_i \psi_j \ .$$

Here, as usual $\theta_i := \theta(\partial_i)$, $\psi_j := \psi(\partial_j)$.

Next, one can define a map $\partial_i \otimes dx^j$ which is linear on forms in the first slot, and linear in vectors in the second slot as

$$(\partial_i \otimes dx^j)(\theta, X) := \partial_i(\theta) dx^j(X) = \theta_i X^j \ . \tag{2.2.10}$$

The $\partial_i \otimes dx^j$'s form the basis for the *space of tensors of rank* $(1,1)$, or *valence* $(1,1)$; if T is such a tensor, then in local coordinates it can be written as

$$T = T^i{}_j \partial_i \otimes dx^j \ .$$

Such a tensor transforms under coordinate changes in the obvious way:

$$T^i{}_j \underbrace{\partial_{x^i}}_{\frac{\partial y^j}{\partial x^i}\partial_{y^j}} \otimes \underbrace{dx^j}_{\frac{\partial x^j}{\partial y^k} dy^k} = T^i{}_j \frac{\partial y^j}{\partial x^i} \frac{\partial x^j}{\partial y^k} \partial_{y^j} \otimes dy^k \ .$$

Quite generally, a tensor of valence, or rank, (r,s) can be defined as an object which has r vector indices and s covector indices, so that it transforms as

$$S^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} \to S^{m_1 \ldots m_r}{}_{\ell_1 \ldots \ell_s} \frac{\partial y^{i_1}}{\partial x^{m_1}} \cdots \frac{\partial y^{i_s}}{\partial x^{m_r}} \frac{\partial x^{\ell_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{\ell_s}}{\partial y^{j_s}}$$

For example, if X^i and Y^j are vectors, then $X^i Y^j$ forms a contravariant tensor of valence two.

Tensors of the same valence can be added in the obvious way: *e.g.*

$$(A+B)(X,Y) := A(X,Y) + B(X,Y) \quad \Longleftrightarrow \quad (A+B)_{ij} = A_{ij} + B_{ij} \ .$$

Tensors can be multiplied by scalars: *e.g.*

$$(fA)(X,Y,Z) := fA(X,Y,Z) \quad \Longleftrightarrow \quad f(A_{ijk}) := (fA_{ijk}) \ .$$

Finally, we have seen in (2.2.8) how to take tensor products for one forms, and in (2.2.10) how to take a tensor product of a vector and a one form, but this can also be done for higher order tensors: e.g., if S is of valence (a,b) and T is a multilinear map of valence (c,d) (not to be confused with the signature!), then $S \otimes T$ is a multilinear map of valence $(a+c, b+d)$, defined as

$$(S \otimes T)(\underbrace{\theta, \ldots \qquad\qquad}_{a \text{ covectors and } b \text{ vectors}}, \underbrace{\psi, \ldots \qquad\qquad}_{c \text{ covectors and } d \text{ vectors}}) := S(\theta, \ldots) T(\psi, \ldots) \ .$$

A rather shorter way of saying what we said so far about tensor products, which includes all the cases we treated separately, is: suppose that V and W are vector spaces, and let α be a p-linear map on V, and β be a q-linear map on W, then, for $X_1, \ldots, X_p \in V$ and $Y_1, \ldots, Y_q \in W$, one defines

$$(\alpha \otimes \beta)(X_1, \ldots, X_p, Y_1, \ldots, Y_q) := \alpha(X_1, \ldots, X_p) \beta(Y_1, \ldots, Y_q) \ .$$

Contractions

The simplest example of the *contraction* applies to tensor fields $S^i{}_j$ with one index down and one index up. One can then perform the sum

$$S^i{}_i \ .$$

This turns out to be a scalar; indeed, under changes of coordinates,

$$S^i{}_j \to \bar{S}^\ell{}_k = S^i{}_j \frac{\partial x^j}{\partial y^k} \frac{\partial y^\ell}{\partial x^i}$$

and

$$\bar{S}^\ell{}_\ell = S^i{}_j \underbrace{\frac{\partial x^j}{\partial y^\ell} \frac{\partial y^\ell}{\partial x^i}}_{\delta^j_i} = S^i{}_i$$

One can similarly do contractions on higher valence tensors, e.g.

$$S^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 j_3 \ldots j_s} \to S^{\ell i_2 \ldots i_r}{}_{j_1 \ell j_3 \ldots j_s}$$

After contraction, a tensor of rank $(r+1, s+1)$ becomes of rank (r, s).

2.2.6 Raising and lowering of indices

•2.2.2 Let g be a symmetric two-covariant tensor field on M, by definition such an object is the assignment to each point $p \in M$ of a bilinear map $g(p)$ from $T_pM \times T_pM$ to $\mathbb{R}$, with the additional property

•2.2.2: this is the right way of looking at the raising and lowering of indices, for those who are interested. The approach described in this section is not examinable, you are only supposed to know formulae such as (2.2.12), or its inverse, the boxed equation in (2.2.13)

$$g(X, Y) = g(Y, X) \ .$$

In this work the symbol g will be reserved to *non-degenerate* symmetric two-covariant tensor fields. It is usual to simply write g for $g(p)$, the point p being implicitly understood. We will sometimes write g_p for $g(p)$ when referencing p.

The usual Sylvester's inertia theorem tells us that at each p the map g will have a well defined signature; clearly this signature will be point-independent on a connected manifold when g is non-degenerate. A pair (M, g) is said to be a *Riemannian manifold* when the signature of g is $(\dim M, 0)$; equivalently, when g is a positive definite bilinear form on every product $T_pM \times T_pM$. A pair (M, g) is said to be a *Lorentzian manifold* when the signature of g is $(\dim M - 1, 1)$. One talks about *pseudo-Riemannian* manifolds whatever the signature of g, as long as g is non-degenerate, but we will only encounter Riemannian and Lorentzian metrics in this work.

Any pseudo-Riemannian g induces an isomorphism

$$\flat : T_pM \to T_p^*M$$

by the formula

$$\boxed{X_\flat(Y) = g(X,Y)}\ .$$

In local coordinates this gives

$$X_\flat = g_{ij}X^i dx^j =: X_j dx^j\ . \tag{2.2.11}$$

This last equality defines X_j — "the vector X^j with the index j lowered":

$$\boxed{X_i := g_{ij}X^j}\ . \tag{2.2.12}$$

The operation (2.2.12) is called the *lowering of indices* in the physics literature and, again in the physics literature, one does not make a distinction between the one-form $X_\flat$ and the vector X.

The inverse map is often denoted by $\sharp$ and is called the *raising of indices*; from (2.2.11) we obviously have

$$\alpha^\sharp = g^{ij}\alpha_i\partial_j =: \alpha^i\partial_i \quad \Longleftrightarrow \quad dx^i(\alpha^\sharp) = \boxed{\alpha^i = g^{ij}\alpha_j}\ , \tag{2.2.13}$$

where (g^{ij}) is the matrix inverse to (g_{ij}). For example,

$$(dx^i)^\sharp = g^{ik}\partial_k\ .$$

Clearly (g^{ij}), understood as the matrix of a bilinear map on T_p^*M, has the same signature as g, and can be used to define a scalar product $g^\sharp$ on $T_p^*(M)$:

$$g^\sharp(\alpha,\beta) := g(\alpha^\sharp,\beta^\sharp) \quad \Longleftrightarrow \quad g^\sharp(dx^i,dx^j) = g^{ij}\ .$$

This last equality is justified as follows:

$$g^\sharp(dx^i,dx^j) = g((dx^i)^\sharp,(dx^j)^\sharp) = g(g^{ik}\partial_k, g^{j\ell}\partial_\ell) = \underbrace{g^{ik}g_{k\ell}}_{=\delta^i_\ell} g^{j\ell} = g^{ji} = g^{ij}\ .$$

It is convenient to use the same letter g for $g^\sharp$ — physicists do it all the time — or for scalar products induced by g on all the remaining tensor bundles, and we will often do so.

One might wish to check by direct calculations that $g_{\mu\nu}X^\nu$ transforms as a one-form if X^μ transforms as a vector. The simplest way is to notice that $g_{\mu\nu}X^\nu$ is a contraction, over the last two indices, of the three-index tensor $g_{\mu\nu}X^\alpha$. Hence it is a one-form by the analysis at the end of the previous section. Alternatively, if we write $\bar g_{\mu\nu}$ for the transformed $g_{\mu\nu}$'s, and $\bar X^\alpha$ for the transformed X^α's, then

$$\underbrace{\bar g_{\alpha\beta}}_{g_{\mu\nu}\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}} \bar X^\beta = g_{\mu\nu}\frac{\partial x^\mu}{\partial y^\alpha}\underbrace{\frac{\partial x^\nu}{\partial y^\beta}\bar X^\beta}_{X^\nu} = g_{\mu\nu}X^\nu\frac{\partial x^\mu}{\partial y^\alpha}\ ,$$

which is indeed the transformation law of a covector.

2.3 Covariant derivatives

When dealing with $\mathbb{R}^n$, or subsets thereof, there exists an obvious prescription for how to differentiate tensor fields: in this case we have at our disposal the canonical "trivialization $\{\partial_i\}_{i=1,\ldots,n}$ of $T\mathbb{R}^n$" (this means: a globally defined set of vectors which, at every point, form a basis of the tangent space), together with its dual trivialization $\{dx^j\}_{i=1,\ldots,n}$ of $T^*\mathbb{R}^n$. We can expand a tensor field T of valence (k,ℓ) in terms of those bases,

$$
\begin{aligned}
T = T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell}\partial_{i_1}\otimes\ldots\otimes\partial_{i_k}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_\ell} \\
\iff \quad T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell} = T(dx^{i_1},\ldots,dx^{i_k},\partial_{j_1},\ldots,\partial_{j_\ell})\ ,
\end{aligned}
\tag{2.3.1}
$$

and differentiate each component $T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell}$ of T separately:

$$
X(T)_{\text{in the coordinate system } x^i} := X^i\partial_{x^i}(T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell})\partial_{x^{i_1}}\otimes\ldots\otimes\partial_{x^{i_k}}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_\ell}\ . \tag{2.3.2}
$$

The resulting object does, however, *not* behave as a tensor under coordinate transformations, in the sense that the above form of the right-hand-side will not be preserved under coordinate transformations: as an example, consider the one-form $T = dx$ on $\mathbb{R}^n$, which has vanishing derivative as defined by (2.3.2). When expressed in spherical coordinates we have

$$
T = d(\rho\cos\varphi) = -\rho\sin\varphi d\varphi + \cos\varphi d\rho\ ,
$$

the partial derivatives of which are non-zero (both with respect to the original cartesian coordinates (x,y) and to the new spherical ones (ρ,φ)). The notion of a *covariant derivative*, sometimes also referred to as a *connection*, is introduced precisely to obtain a notion of derivative which has tensorial properties. By definition, a covariant derivative is a map which to a vector field X and a tensor field T assigns a tensor field of the same type as T, denoted by $\nabla_X T$, with the following properties:

1. $\nabla_X T$ is linear with respect to addition both with respect to X and T:

$$
\nabla_{X+Y}T = \nabla_X T + \nabla_Y T\ , \qquad \nabla_X(T+Y) = \nabla_X T + \nabla_X Y\ ; \tag{2.3.3}
$$

2. $\nabla_X T$ is linear with respect to multiplication of X by functions f,

$$
\nabla_{fX}T = f\nabla_X T\ ; \tag{2.3.4}
$$

3. and, finally, $\nabla_X T$ satisfies the *Leibniz rule* under multiplication of T by a differentiable function f:

$$
\nabla_X(fT) = f\nabla_X T + X(f)T\ . \tag{2.3.5}
$$

It is natural to ask whether covariant derivatives do exist at all in general and, if so, how many of them can there be. First, it immediately follows from the axioms above that if D and ∇ are two covariant derivatives, then

$$\Delta(X,T) := D_X T - \nabla_X T$$

is multi-linear both with respect to addition and multiplication by functions — the non-homogeneous terms $X(f)T$ in (2.3.5) cancel — and is thus a tensor field. Reciprocally, if D is a covariant derivative and $\Delta(X,T)$ is bilinear with respect to addition and multiplication by functions, then

$$\nabla_X T := D_X T + \Delta(X,T) \tag{2.3.6}$$

is a new covariant derivative. So, at least locally, on tensors of valence (r,s) there are as many covariant derivatives as tensors of valence $(r+s, r+s+1)$.

Functions

The *canonical covariant derivative on functions* is defined as

$$\nabla_X(f) = X(f) ,$$

and we will always use the above. This has all the right properties, so obviously covariant derivatives of functions exist. From what has been said, any covariant derivative on functions is of the form

$$\nabla_X f = X(f) + \alpha(X)f , \tag{2.3.7}$$

where α is a one-form. Conversely, given any one form α, (2.3.7) defines a covariant derivative on functions. The addition of the lower-order term $\alpha(X)f$ in (2.3.7) does not appear to be very useful for functions, but it plays a role in a geometric formulation of electrodynamics, or in geometric quantization. In any case such lower-order terms play an essential role when defining covariant derivatives for tensor fields.

Vectors

The simplest next possibility is that of a covariant derivative of vector fields. Let us not worry about existence at this stage, but assume that a covariant derivative exists, and work from there. (Anticipating, we will show shortly that any metric defines a covariant derivative, called the *Levi-Civita* covariant derivative, which is the unique covariant derivative operator satisfying a natural set of conditions, to be discussed below.)

We will first assume that we are working on a set $\Omega \subset M$ over which we have a *global trivialization* of the tangent bundle TM; by definition, this means that there exist vector fields e_a, $a = 1, \ldots, \dim M$, such that at every point $p \in \Omega$ the fields $e_a(p) \in T_pM$ form a basis of T_pM.[1] •2.3.1

•2.3.1: *general bases are not examinable; you are welcome to work with them if you wish, but for the purposes of this course you can think of* $\{e_a\}$ *as being just another way of writing* $\{\partial_a\}$*, if you are more comfortable with that*

Let θ^a denote the dual trivialization of T^*M — by definition the θ^a's satisfy

$$\boxed{\theta^a(e_b) = \delta^a_b} \,.$$

Given a covariant derivative ∇ on vector fields we set

$$\Gamma^a{}_b(X) := \theta^a(\nabla_X e_b) \quad \Longleftrightarrow \quad \nabla_X e_b = \Gamma^a{}_b(X) e_a \,, \tag{2.3.8a}$$

$$\boxed{\Gamma^a{}_{bc} := \Gamma^a{}_b(e_c) = \theta^a(\nabla_{e_c} e_b)} \quad \Longleftrightarrow \quad \nabla_X e_b = \Gamma^a{}_{bc} X^c e_a \,. \tag{2.3.8b}$$

The (locally defined) functions $\Gamma^a{}_{bc}$ are called *connection coefficients*. If $\{e_a\}$ is the coordinate basis $\{\partial_\mu\}$ we shall write

$$\Gamma^\mu{}_{\alpha\beta} := dx^\mu(\nabla_{\partial_\beta}\partial_\alpha) \quad \Big(\Longleftrightarrow \quad \nabla_{\partial_\mu}\partial_\nu = \Gamma^\sigma{}_{\nu\mu}\partial_\sigma \Big) \,, \tag{2.3.9}$$

etc. In this particular case the connection coefficients are usually called *Christoffel symbols*. We will sometimes write $\Gamma^\sigma_{\nu\mu}$ instead of $\Gamma^\sigma{}_{\nu\mu}$; note that the latter convention is more common. By using the Leibniz rule (2.3.5) we find

$$\begin{aligned}
\nabla_X Y &= \nabla_X(Y^a e_a) \\
&= X(Y^a)e_a + Y^a \nabla_X e_a \\
&= X(Y^a)e_a + Y^a \Gamma^b{}_a(X) e_b \\
&= (X(Y^a) + \Gamma^a{}_b(X) Y^b) e_a \\
&= (X(Y^a) + \Gamma^a{}_{bc} Y^b X^c) e_a \,,
\end{aligned} \tag{2.3.10}$$

which gives various equivalent ways of writing $\nabla_X Y$. The (perhaps only locally defined) $\Gamma^a{}_b$'s are linear in X, and the collection $(\Gamma^a{}_b)_{a,b=1,\ldots,\dim M}$ is sometimes referred to as the *connection one-form*.

We will often write ∇_a for ∇_{e_a}.

The one-covariant, one-contravariant tensor field ∇Y is defined as

$$\nabla Y := \nabla_a Y^b \theta^a \otimes e_b \Longleftrightarrow \nabla_a Y^b := \theta^b(\nabla_{e_a} Y) \Longleftrightarrow \boxed{\nabla_a Y^b = e_a(Y^b) + \Gamma^b{}_{ca} Y^c} \,. \tag{2.3.11}$$

$\nabla_a Y^b$ will sometimes be written as $Y^b{}_{;a}$.

[1] This is the case when Ω is a coordinate patch with coordinates (x^a), then the $\{e_a\}_{a=1,\ldots,\dim M}$ can be chosen to be equal to $\{\partial_a\}_{a=1,\ldots,\dim M}$. Recall that a manifold is said to be parallelizable if a basis of TM can be chosen globally over M — in such a case Ω can be taken equal to M. We emphasize that we are *not* assuming that M is parallelizable, so that equations such as (2.3.8) have only a local character in general.

Transformation law

Consider a coordinate basis ∂_{x^i}, it is natural to enquire about the transformation law of the connection coefficients $\Gamma^i{}_{jk}$ under a change of coordinates $x^i \to y^k(x^i)$. To make things clear, let us write $\Gamma^i{}_{jk}$ for the connection coefficients in the x–coordinates, and $\hat\Gamma^i{}_{jk}$ for the ones in the y–cordinates. We calculate:

$$\begin{aligned}
\Gamma^i{}_{jk} &:= dx^i\Big(\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^j}\Big)\\
&= dx^i\Big(\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial y^\ell}{\partial x^j}\frac{\partial}{\partial y^\ell}\Big)\\
&= dx^i\Big(\frac{\partial^2 y^\ell}{\partial x^k\partial x^j}\frac{\partial}{\partial y^\ell}+\frac{\partial y^\ell}{\partial x^j}\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial y^\ell}\Big)\\
&= \frac{\partial x^i}{\partial y^s}dy^s\Big(\frac{\partial^2 y^\ell}{\partial x^k\partial x^j}\frac{\partial}{\partial y^\ell}+\frac{\partial y^\ell}{\partial x^j}\nabla_{\frac{\partial y^r}{\partial x^k}\frac{\partial}{\partial y^r}}\frac{\partial}{\partial y^\ell}\Big)\\
&= \frac{\partial x^i}{\partial y^s}dy^s\Big(\frac{\partial^2 y^\ell}{\partial x^k\partial x^j}\frac{\partial}{\partial y^\ell}+\frac{\partial y^\ell}{\partial x^j}\frac{\partial y^r}{\partial x^k}\nabla_{\frac{\partial}{\partial y^r}}\frac{\partial}{\partial y^\ell}\Big)\\
&= \frac{\partial x^i}{\partial y^s}\frac{\partial^2 y^s}{\partial x^k\partial x^j}+\frac{\partial x^i}{\partial y^s}\frac{\partial y^\ell}{\partial x^j}\frac{\partial y^r}{\partial x^k}\hat\Gamma^s{}_{\ell r}\ .
\end{aligned} \tag{2.3.12}$$

Summarising,

$$\boxed{\Gamma^i{}_{jk} = \hat\Gamma^s{}_{\ell r}\frac{\partial x^i}{\partial y^s}\frac{\partial y^\ell}{\partial x^j}\frac{\partial y^r}{\partial x^k}+\frac{\partial x^i}{\partial y^s}\frac{\partial^2 y^s}{\partial x^k\partial x^j}}\ . \tag{2.3.13}$$

Thus, the $\Gamma^i{}_{jk}$'s do *not* form a tensor; instead they transform as a tensor *plus* a non-homogeneous term containing second derivatives, as seen above.

Torsion

Because the inhomogeneous term in (2.3.13) is symmetric under the interchange of i and j, it follows from (2.3.13) that

$$T^i_{jk} := \Gamma^i{}_{kj} - \Gamma^i{}_{jk}$$

does transform as a tensor, called *the torsion tensor* of ∇.

An index-free definition of torsion proceeds as follows: Let ∇ be a covariant derivative defined for vector fields, the *torsion tensor* T is defined by the formula

$$\boxed{T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]}\ , \tag{2.3.14}$$

where $[X,Y]$ is the Lie bracket. We obviously have

$$T(X,Y) = -T(Y,X)\ . \tag{2.3.15}$$

Let us check that T is actually a tensor field: multi-linearity with respect to addition is obvious. To check what happens under multiplication by functions, in view of (2.3.15) it is sufficient to do the calculation for the first slot of T. We then have

$$\begin{aligned} T(fX,Y) &= \nabla_{fX}Y - \nabla_Y(fX) - [fX,Y] \\ &= f\Big(\nabla_X Y - \nabla_Y X\Big) - Y(f)X - [fX,Y]\,. \end{aligned} \tag{2.3.16}$$

To work out the last commutator term we compute, for any function g,

$$[fX,Y](g) = fX(Y(g)) - \underbrace{Y(fX(g))}_{=Y(f)X(g)+fY(X(g))} = f[X,Y](g) - Y(f)X(g)\ ,$$

hence

$$[fX,Y] = f[X,Y] - Y(f)X\ , \tag{2.3.17}$$

and the last term here cancels the undesirable second-to-last term in (2.3.16), as required.

In a coordinate basis ∂_μ we have $[\partial_\mu, \partial_\nu] = 0$ and one finds from (2.3.9)

$$\boxed{T_{\mu\nu} := T(\partial_\mu, \partial_\nu) = (\Gamma^\sigma{}_{\nu\mu} - \Gamma^\sigma{}_{\mu\nu})\partial_\sigma}\ , \tag{2.3.18}$$

which shows that — in coordinate frames — T is determined by twice the antisymmetrization of the $\Gamma^\sigma{}_{\mu\nu}$'s over the lower indices. In particular, as already noticed by inspection of the transformation law for the Christoffel symbols, that last antisymmetrization produces a tensor field.

Covectors

Suppose that we are given a covariant derivative on vector fields, there is a natural way of inducing a covariant derivative on one-forms by imposing the condition that *the duality operation be compatible with the Leibniz rule*: given two vector fields X and Y together with a field of one-forms α, one sets

$$\boxed{(\nabla_X\alpha)(Y) := X(\alpha(Y)) - \alpha(\nabla_X Y)}\,. \tag{2.3.19}$$

Let us, first, check that (2.3.19) indeed defines a field of one-forms. The linearity, in the Y variable, with respect to addition is obvious. Next, for any function f we have

$$\begin{aligned} (\nabla_X\alpha)(fY) &= X(\alpha(fY)) - \alpha(\nabla_X(fY)) \\ &= X(f)\alpha(Y) + fX(\alpha(Y)) - \alpha(X(f)Y + f\nabla_X Y) \\ &= f(\nabla_X\alpha)(Y)\ , \end{aligned}$$

as should be the case for one-forms. Further, we need to check that ∇ defined by (2.3.19) does satisfy the remaining axioms imposed on covariant derivatives.

Again multi-linearity with respect to addition is obvious, as well as linearity with respect to multiplication of X by a function. Finally,

$$\begin{aligned}\nabla_X(f\alpha)(Y) &= X(f\alpha(Y)) - f\alpha(\nabla_X Y)\\ &= X(f)\alpha(Y) + f(\nabla_X\alpha)(Y)\ ,\end{aligned}$$

as desired.

The duality pairing

$$T_p^*M \times T_pM \ni (\alpha, X) \to \alpha(X) \in \mathbb{R}$$

is sometimes called *contraction*. As already pointed out, the operation ∇ on one forms has been defined in (2.3.19) so as to satisfy the *Leibniz rule under duality pairing*:

$$X(\alpha(Y)) = (\nabla_X\alpha)(Y) + \alpha(\nabla_X Y)\ ; \tag{2.3.20}$$

this follows directly from (2.3.19). This should not be confused with the Leibniz rule under multiplication by functions, which is part of the definition of a covariant derivative, and therefore always holds. It should be kept in mind that (2.3.20) does not necessarily hold for all covariant derivatives: if ${}^v\nabla$ is some covariant derivative on vectors, and ${}^f\nabla$ is some covariant derivative on one-forms, in general one will have

$$X(\alpha(Y)) \neq ({}^f\nabla_X)\alpha(Y) + \alpha({}^v\nabla_X Y)\ .$$

Using the basis-expression (2.3.10) of $\nabla_X Y$ and the definition (2.3.19) we have

$$\nabla_X\alpha = X^a\nabla_a\alpha_b\,\theta^b\ ,$$

with

$$\begin{aligned}\boxed{\nabla_a\alpha_b} &:= (\nabla_{e_a}\alpha)(e_b)\\ &= e_a(\alpha(e_b)) - \alpha(\nabla_{e_a}e_b)\\ &= \boxed{e_a(\alpha_b) - \Gamma^c{}_{ba}\alpha_c}\ .\end{aligned}$$

Higher order tensors

It should now be clear how to extend ∇ to tensors of arbitrary valence: if T is r covariant and s contravariant one sets

$$\begin{aligned}(\nabla_X T)(X_1,\ldots,X_r,\alpha_1,\ldots\alpha_s) &:= X\Big(T(X_1,\ldots,X_r,\alpha_1,\ldots\alpha_s)\Big)\\ &-T(\nabla_X X_1,\ldots,X_r,\alpha_1,\ldots\alpha_s) - \ldots - T(X_1,\ldots,\nabla_X X_r,\alpha_1,\ldots\alpha_s)\\ &-T(X_1,\ldots,X_r,\nabla_X\alpha_1,\ldots\alpha_s) - \ldots - T(X_1,\ldots,X_r,\alpha_1,\ldots\nabla_X\alpha_s)\ .\end{aligned} \tag{2.3.21}$$

The verification that this defines a covariant derivative proceeds in a way identical to that for one-forms. In a basis we have

$$\nabla_X T = X^a \nabla_a T_{a_1\ldots a_r}{}^{b_1\ldots b_s} \theta^{a_1} \otimes \ldots \otimes \theta^{a_r} \otimes e_{b_1} \otimes \ldots \otimes e_{b_s} \ ,$$

and (2.3.21) gives

$$\begin{aligned} \nabla_a T_{a_1\ldots a_r}{}^{b_1\ldots b_s} &:= (\nabla_{e_a} T)(e_{a_1}, \ldots, e_{a_r}, \theta^{b_1}, \ldots, \theta^{b_s}) \\ &= e_a(T_{a_1\ldots a_r}{}^{b_1\ldots b_s}) - \Gamma^c{}_{a_1 a} T_{c\ldots a_r}{}^{b_1\ldots b_s} - \ldots - \Gamma^c{}_{a_r a} T_{a_1\ldots c}{}^{b_1\ldots b_s} \\ &\quad + \Gamma^{b_1}{}_{ca} T_{a_1\ldots a_r}{}^{c\ldots b_s} + \ldots + \Gamma^{b_s}{}_{ca} T_{a_1\ldots a_r}{}^{b_1\ldots c} \ . \end{aligned} \tag{2.3.22}$$

Carrying over the last two lines of (2.3.21) to the left-hand-side of that equation one obtains the Leibniz rule for ∇ under pairings of tensors with vectors or forms. It should be clear from (2.3.21) that ∇ defined by that equation is the *only covariant derivative which agrees with the original one on vectors, and which satisfies the Leibniz rule under the pairing operation.* We will only consider such covariant derivatives in this work.

2.3.1 The Levi-Civita connection

One of the fundamental results in pseudo-Riemannian geometry is that of the existence of a torsion-free connection which preserves the metric:

THEOREM 2.3.1 *Let g be a two-covariant symmetric non-degenerate tensor field on a manifold M. Then there exists a unique connection ∇ such that*

1. *$\nabla g = 0$,*
2. *the torsion tensor T of ∇ vanishes.*

PROOF: Using the definition of $\nabla_i g_{jk}$ we have

$$0 = \nabla_i g_{jk} \equiv \partial_i g_{jk} - \Gamma^\ell_{ji} g_{\ell k} - \Gamma^\ell_{ki} g_{\ell j} \ ; \tag{2.3.23}$$

here we have written Γ^i_{jk} instead of $\Gamma^i{}_{jk}$, as is standard in the literature. We rewrite this equation making cyclic permutations of indices, and changing the overall sign:

$$0 = -\nabla_j g_{ki} \equiv -\partial_j g_{ki} + \Gamma^\ell_{kj} g_{\ell i} + \Gamma^\ell_{ij} g_{\ell k} \ .$$

$$0 = -\nabla_k g_{ij} \equiv -\partial_k g_{ij} + \Gamma^\ell_{ik} g_{\ell j} + \Gamma^\ell_{jk} g_{\ell i} \ .$$

Adding the three equations and using symmetry of Γ^k_{ji} in ij one obtains

$$0 = \partial_i g_{jk} - \partial_j g_{ki} - \partial_k g_{ij} + 2\Gamma^\ell_{jk} g_{\ell i} \ ,$$

Multiplying by g^{im} we obtain

$$\Gamma^m_{jk} = g^{mi} \Gamma^\ell_{jk} g_{\ell i} = \frac{1}{2} g^{mi} (\partial_i g_{jk} - \partial_j g_{ki} - \partial_k g_{ij}) \ . \tag{2.3.24}$$

This proves uniqueness.

It then remains to check that the insertion of Γ^m_{jk}, as given by (2.3.24), into the right-hand-side of (2.3.23), indeed gives zero, proving existence.

Let us give a coordinate-free version of the above, which turns out to be much messier: Suppose, first, that a connection satisfying the above is given. By the Leibniz rule we then have for any vector fields X, Y and Z,

$$0 = (\nabla_X g)(Y,Z) = X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) . \tag{2.3.25}$$

One then rewrites the same equation applying cyclic permutations to X, Y, and Z, with a minus sign for the last equation: •2.3.2

•2.3.2: *If in the next two equations one takes $X = \partial_i$, $Y = \partial_j$ and $Z = \partial_k$, then $[\partial_i, \partial_j] = 0$ so many terms drop out; the proof using indices, admittedly much simpler, can be found at the end of the current section*

$$\begin{aligned} g(\nabla_X Y, Z) + g(Y, \nabla_X Z) &= X(g(Y,Z)) ,\\ g(\nabla_Y Z, X) + g(Z, \nabla_Y X) &= Y(g(Z,X)) ,\\ -g(\nabla_Z X, Y) - g(X, \nabla_Z Y) &= -Z(g(X,Y)) . \end{aligned} \tag{2.3.26}$$

As the torsion tensor vanishes, the sum of the left-hand-sides of these equations can be manipulated as follows:

$$\begin{aligned} & g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + g(\nabla_Y Z, X) + g(Z, \nabla_Y X) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)\\ &= g(\nabla_X Y + \nabla_Y X, Z) + g(Y, \nabla_X Z - \nabla_Z X) + g(X, \nabla_Y Z - \nabla_Z Y)\\ &= g(2\nabla_X Y - [X,Y], Z) + g(Y, [X,Z]) + g(X, [Y,Z])\\ &= 2g(\nabla_X Y, Z) - g([X,Y],Z) + g(Y,[X,Z]) + g(X,[Y,Z]) . \end{aligned}$$

This shows that the sum of the three equations (2.3.26) can be rewritten as

$$\begin{aligned} 2g(\nabla_X Y, Z) &= g([X,Y],Z) - g(Y,[X,Z]) - g(X,[Y,Z])\\ &\quad + X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) . \end{aligned} \tag{2.3.27}$$

Since Z is arbitrary and g is non-degenerate, the left-hand-side of this equation determines the vector field $\nabla_X Y$ uniquely, and uniqueness of ∇ follows.

To prove existence, let $S(X,Y)(Z)$ be defined as one half of the right-hand-side of (2.3.27),

$$\begin{aligned} S(X,Y)(Z) &= \frac{1}{2}\Big(X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))\\ &\quad + g(Z,[X,Y]) - g(Y,[X,Z]) - g(X,[Y,Z])\Big) . \end{aligned} \tag{2.3.28}$$

Clearly S is linear with respect to addition in all fields involved. It is straightforward to check that it is linear with respect to multiplication of Z by a function, and since g is non-degenerate there exists a unique vector field $W(X,Y)$ such that

$$S(X,Y)(Z) = g(W(X,Y),Z) .$$

One readily checks that the assignment

$$(X,Y) \to W(X,Y)$$

satisfies all the requirements imposed on a covariant derivative $\nabla_X Y$. With some more work one checks that ∇_X so defined is torsion free, and metric compatible. □

Let us check that (2.3.27) reproduces (2.3.24): Consider (2.3.27) with $X = \partial_\gamma$, $Y = \partial_\beta$ and $Z = \partial_\sigma$,

$$\begin{aligned} 2g(\nabla_\gamma \partial_\beta, \partial_\sigma) &= 2g(\Gamma^\rho{}_{\beta\gamma}\partial_\rho, \partial_\sigma) \\ &= 2g_{\rho\sigma}\Gamma^\rho{}_{\beta\gamma} \\ &= \partial_\gamma g_{\beta\sigma} + \partial_\beta g_{\gamma\sigma} - \partial_\sigma g_{\beta\gamma} \end{aligned} \tag{2.3.29}$$

Multiplying this equation by $g^{\alpha\sigma}/2$ we then obtain

$$\boxed{\Gamma^\alpha{}_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\sigma}\{\partial_\beta g_{\sigma\gamma} + \partial_\gamma g_{\sigma\beta} - \partial_\sigma g_{\beta\gamma}\}}\ . \tag{2.3.30}$$

2.3.2 Geodesics and Christoffel symbols

A geodesic can be defined as the stationary point of the action

$$I(\gamma) = \frac{1}{2}\int_a^b \underbrace{g(\dot\gamma, \dot\gamma)(s)}_{=:\mathscr{L}(\gamma,\dot\gamma)} ds\ , \tag{2.3.31}$$

where $\gamma : [a, b] \to M$ is a differentiable curve. Thus,

$$\mathscr{L}(x^\mu, \dot x^\nu) = \frac{1}{2} g_{\alpha\beta}(x^\mu)\dot x^\alpha \dot x^\beta\ .$$

One readily finds the Euler-Lagrange equations for this Lagrange function: •2.3.3

•2.3.3: a detailed calculation can be found in Woodhouse's book

$$\frac{d}{ds}\left(\frac{\partial \mathscr{L}}{\partial \dot x^\mu}\right) = \frac{\partial \mathscr{L}}{\partial x^\mu} \quad \Longleftrightarrow \quad \frac{d^2 x^\mu}{ds^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0\ . \tag{2.3.32}$$

This gives a very convenient way of calculating the Christoffel symbols: given a metric g, write down $\mathscr{L}$, work out the Euler-Lagrange equations, and identify the Christoffels as the coefficients of the first derivative terms in those equations.

(The Euler-Lagrange equations for (2.3.31) are identical with those of

$$\tilde I(\gamma) = \int_a^b \sqrt{|g(\dot\gamma, \dot\gamma)(s)|}ds\ , \tag{2.3.33}$$

but (2.3.31) is more convenient to work with. For example, $\mathscr{L}$ is differentiable at points where $\dot\gamma$ vanishes, while $\sqrt{|g(\dot\gamma, \dot\gamma)(s)|}$ is not. The aesthetic advantage of (2.3.33), of being reparameterization-invariant, is more than compensated by the calculational convenience of $\mathscr{L}$.)

Example 2.3.2 As an example, consider a metric of the form

$$g = dr^2 + f(r)d\varphi^2\ .$$

Special cases of this metric include the Euclidean metric on $\mathbb{R}^2$ (then $f(r) = r^2$), and the canonical metric on a sphere (then $f(r) = \sin^2 r$, with r actually being the polar angle θ). The Lagrangian (2.3.33) is thus

$$L = \frac{1}{2}\left(\dot r^2 + f(r)\dot\varphi^2\right) .$$

The Euler-Lagrange equations read

$$\underbrace{\frac{\partial L}{\partial \varphi}}_{0} = \frac{d}{ds}\left(\frac{\partial L}{\partial \dot\varphi}\right) = \frac{d}{ds}(f(r)\dot\varphi) ,$$

so that

$$0 = f\ddot\varphi + f'\dot r\dot\varphi = f\left(\ddot\varphi + \Gamma^\varphi_{\varphi\varphi}\dot\varphi^2 + 2\Gamma^\varphi_{r\varphi}\dot r\dot\varphi + \Gamma^\varphi_{r}\dot r^2\right) \quad\Longrightarrow\quad \Gamma^\varphi_{\varphi\varphi} = \Gamma^\varphi_{rr} = 0 , \quad \Gamma^\varphi_{r\varphi} = \frac{f'}{2f} .$$

Similarly

$$\underbrace{\frac{\partial L}{\partial r}}_{f'\dot\varphi^2/2} = \frac{d}{ds}\left(\frac{\partial L}{\partial \dot r}\right) = \ddot r ,$$

so that

$$\Gamma^r_{r\varphi} = \Gamma^r_{rr} = 0 , \quad \Gamma^r_{\varphi\varphi} = -\frac{f'}{2} .$$

2.4 Local inertial coordinates

Proposition 2.4.1 *1. Let g be a Lorentzian metric, for every $p \in M$ there exists a neighborhood thereof with a coordinate system such that $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1, -1, \cdots, -1)$ at p.*

2. If g is differentiable, then the coordinates can be further chosen so that

$$\partial_\sigma g_{\alpha\beta} = 0 \tag{2.4.1}$$

at p.

Proof: 1. Let y^μ be any coordinate system around p, shifting by a constant vector we can assume that p corresponds to $y^\mu = 0$. Let $e_a = e_a{}^\mu \partial/\partial y^\mu$ be any frame at p such that $g(e_a, e_b) = \eta_{ab}$ — such frames can be found by, *e.g.*, a Gram-Schmidt orthogonalisation. Calculating the determinant of both sides of the equation

$$g_{\mu\nu} e_a{}^\mu e_b{}^\nu = \eta_{ab}$$

we obtain, at p,

$$\det(g_{\mu\nu}) \det(e_a{}^\mu)^2 = -1 ,$$

which shows that $\det(e_a{}^\mu)$ is non-vanishing. It follows that the formula

$$y^\mu = e^\mu{}_a x^a$$

defines a (linear) diffeomorphism. In the new coordinates we have, again at p,

$$g\Big(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\Big) = e^\mu{}_a e^\nu{}_b g\Big(\frac{\partial}{\partial y^\mu}, \frac{\partial}{\partial y^\nu}\Big) = \eta_{ab} \ . \tag{2.4.2}$$

2. We will use (2.3.13), which uses latin indices, so let us switch to that notation. Let x^i be the coordinates described in point 1., recall that p lies at the origin of those coordinates. The new coordinates $\hat{x}^j$ will be implicitly defined by the equations

$$x^i = \hat{x}^i + \frac{1}{2} A^i{}_{jk} \hat{x}^j \hat{x}^k \ ,$$

where $A^i{}_{jk}$ is a set of constants, symmetric with respect to the interchange of j and k. Recall (2.3.13),

$$\hat{\Gamma}^i{}_{jk} = \Gamma^s{}_{\ell r} \frac{\partial \hat{x}^i}{\partial x^s} \frac{\partial x^\ell}{\partial \hat{x}^j} \frac{\partial x^r}{\partial \hat{x}^k} + \frac{\partial \hat{x}^i}{\partial x^s} \frac{\partial^2 x^s}{\partial \hat{x}^k \partial \hat{x}^j} \ ; \tag{2.4.3}$$

here we use $\hat{\Gamma}^s_{\ell r}$ to denote the Christoffel symbols of the metric in the hatted coordinates. Then, at $x^i = 0$, this equation reads

$$\begin{aligned} \hat{\Gamma}^i{}_{jk} &= \Gamma^s{}_{\ell r} \underbrace{\frac{\partial \hat{x}^i}{\partial x^s}}_{\delta^i_s} \underbrace{\frac{\partial x^\ell}{\partial \hat{x}^j}}_{\delta^\ell_j} \underbrace{\frac{\partial x^r}{\partial \hat{x}^k}}_{\delta^r_k} + \underbrace{\frac{\partial x^i}{\partial x^s}}_{\delta^i_s} \underbrace{\frac{\partial^2 x^s}{\partial \hat{x}^k \partial \hat{x}^j}}_{A^s_{kj}} \\ &= \Gamma^i{}_{jk} + A^i{}_k \ . \end{aligned}$$

Choosing A^i_{jk} as $-\Gamma^i{}_{jk}(0)$, the result follows.

If you do not like to remember formulae such as (2.3.13), proceed as follows: Let x^μ be the coordinates described in point 1. The new coordinates $\hat{x}^\alpha$ will be implicitly defined by the equations

$$x^\mu = \hat{x}^\mu + \frac{1}{2} A^\mu{}_{\alpha\beta} \hat{x}^\alpha \hat{x}^\beta \ ,$$

where $A^\mu{}_{\alpha\beta}$ is a set of constants, symmetric with respect to the interchange of α and β. Set

$$\hat{g}_{\alpha\beta} := g\Big(\frac{\partial}{\partial \hat{x}^\alpha}, \frac{\partial}{\partial \hat{x}^\beta}\Big) \ , \qquad g_{\alpha\beta} := g\Big(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\Big) \ .$$

Recall the transformation law

$$\hat{g}_{\mu\nu}(\hat{x}^\sigma) = g_{\alpha\beta}(x^\rho(\hat{x}^\sigma)) \frac{\partial x^\alpha}{\partial \hat{x}^\mu} \frac{\partial x^\beta}{\partial \hat{x}^\nu} \ .$$

By differentiation one obtains at $x^\mu = \hat{x}^\mu = 0$,

$$\begin{aligned}
\frac{\partial \hat{g}_{\mu\nu}}{\partial \hat{x}^\rho}(0) &= \frac{\partial g_{\mu\nu}}{\partial x^\rho}(0) + g_{\alpha\beta}(0)\Big(A^\alpha{}_{\mu\rho}\delta^\beta_\nu + \delta^\alpha_\mu A^\beta{}_{\nu\rho}\Big) \\
&= \frac{\partial g_{\mu\nu}}{\partial x^\rho}(0) + A_{\nu\mu\rho} + A_{\mu\nu\rho} \;, \qquad (2.4.4)
\end{aligned}$$

where

$$A_{\alpha\beta\gamma} := g_{\alpha\sigma}(0)A^\sigma{}_{\beta\gamma} \;.$$

It remains to show that we can choose $A^\sigma{}_{\beta\gamma}$ so that the left-hand-side can be made to vanish at p. An explicit formula for $A_{\sigma\beta\gamma}$ can be obtained from (2.4.4) by a cyclic permutation calculation similar to that in (2.3.26). After raising the first index, the final result is

$$A^\alpha{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\rho}\left\{\frac{\partial g_{\beta\gamma}}{\partial x^\rho} - \frac{\partial g_{\beta\rho}}{\partial x^\gamma} - \frac{\partial g_{\rho\gamma}}{\partial x^\beta}\right\}(0) \;;$$

the reader may wish to check directly that this does indeed lead to a vanishing right-hand-side of (2.4.4).

□

2.5 Curvature

We have seen that we can get rid of first derivatives of the metric at any point by making a coordinate transformation. It turns out that second derivatives cannot be gotten rid of in this way. This fact will follow from the study of a new object, called the curvature tensor.

PROPOSITION 2.5.1 *1. Let ∇ be torsion-free. There exists a tensor field $R_{abc}{}^d$ of type $(1,3)$ such that*

$$\nabla_a\nabla_b X^d - \nabla_b\nabla_a X^d = R_{abc}{}^d X^c \;. \qquad (2.5.1)$$

2. Furthermore,

$$R_{abc}{}^d = \partial_a\Gamma^d_{bc} - \partial_b\Gamma^d{}_{ac} + \Gamma^d_{ae}\Gamma^e_{bc} - \Gamma^d_{be}\Gamma^e_{ac} \;. \qquad (2.5.2)$$

PROOF: We need to check that the derivatives of X cancel. Now,

$$\begin{aligned}
\nabla_a\nabla_b X^d &= \partial_a(\underbrace{\nabla_b X^d}_{\partial_b X^d + \Gamma^d{}_{be}X^e}) + \Gamma^d_{ac}\underbrace{\nabla_b X^c}_{\partial_b X^c + \Gamma^c_{be}X^e} - \Gamma^e_{ab}\nabla_e X^d \\
&= \underbrace{\partial_a\partial_b X^d}_{=:1_{ab}} + \partial_a\Gamma^d{}_{be}X^e + \underbrace{\Gamma^d_{be}\partial_a X^e}_{=:2_{ab}} + \underbrace{\Gamma^d_{ac}\partial_b X^c}_{=:3_{ab}} + \Gamma^d_{ac}\Gamma^c_{be}X^e - \underbrace{\Gamma^e_{ab}\nabla_e X^d}_{=:4_{ab}} \;.
\end{aligned}$$

If we subtract $\nabla_b\nabla_a X^d$, then

1. 1_{ab} is symmetric in a and b, so will cancel out; similarly for 4_{ab} because ∇ has been assumed to have no torsion;

2. 2_{ab} will cancel out with 3_{ba}; similarly 3_{ab} will cancel out with 2_{ba}.

So the left-hand-side of (2.5.1) is indeed linear in X^e. Since it is a tensor, the right-hand-side also is. Since X^e is arbitrary, we conclude that $R_{abc}{}^d$ is a tensor of the desired type. This proves point 1.

To prove 2., from what has been said we have

$$\begin{aligned}\nabla_a\nabla_b X^d - \nabla_b\nabla_a X^d &= \partial_a\Gamma^d_{be}X^e + \Gamma^d_{ac}\Gamma^c_{be}X^e - (a \longleftrightarrow b)\\ &= (\partial_a\Gamma^d_{bc} - \partial_b\Gamma^d{}_{ac} + \Gamma^d_{ae}\Gamma^e_{bc} - \Gamma^d_{be}\Gamma^e_{ac})X^c \;,\end{aligned}$$

as desired. □

The calculation of the curvature tensor may be a very traumatic experience. There is one obvious case where things are painless, when all $g_{\mu\nu}$'s are constants: in this case the Christoffels vanish, and so does the curvature tensor. Metrics with the last property are called *flat*.

For more general metrics, one way out is to use symbolic computer algebra. MATHEMATICA packages to do this can be found at URL's `http://www.math.washington.edu/~lee/Ricci`, or `http://grtensor.phy.queensu.ca/NewDemo`, or `http://luth.obspm.fr/~luthier/Martin-Garcia/xAct`. This last package is least-user-friendly as of today, but is the most flexible, especially for more involved computations.

EXAMPLE 2.5.2 As an example less trivial than a metric with constant coefficients, consider the round two sphere, which we write in the form

$$g = d\theta^2 + e^{2f}d\varphi^2 \;, \qquad e^{2f} = \sin^2\theta \;.$$

The Christoffel symbols are easily found from the Lagrangian for geodesics (see example 2.3.2):

$$\mathscr{L} = \frac{1}{2}(\dot\theta^2 + e^{2f}\dot\varphi^2) \;.$$

The Euler-Lagrange equations give

$$\Gamma^\theta{}_{\varphi\varphi} = -f'e^{2f} \;, \quad \Gamma^\varphi{}_{\theta\varphi} = \Gamma^\varphi{}_{\varphi\theta} = f' \;,$$

with the remaining Christoffel symbols vanishing. Using the definition of the Riemann tensor we then immediately find that

$$R_{\theta\varphi\theta}{}^\varphi = f'' + (f')^2 = -1 \;.$$

All remaining components of the Riemann tensor can be obtained from this one by raising and lowering of indices, together with symmetry operations.

From this one also finds •2.5.1

$$R_{ab} := R_{acb}{}^c = -g_{ab} \;, \qquad R = -2 \;.$$

•2.5.1: note that in other conventions one obtains plus signs here; this is more common in the Riemannian context

•2.5.2 As an exercice, we check that an equivalent, index-free definition can be given as follows: Let ∇ be a *torsionless* covariant derivative defined for vector fields, the definition •2.5.2: *this is not examinable*

$$\boxed{R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z}\ , \tag{2.5.3}$$

where, as elsewhere, $[X,Y]$ is the Lie bracket , which coincides with the one above if one sets

$$R_{abc}{}^d = dx^d(R(\partial_a,\partial_b)\partial_c) \quad\Longleftrightarrow\quad R(\partial_a,\partial_b)\partial_c = R_{abc}{}^d\partial_d\ .$$

To prove this, given a vector field Z, consider the tensor field S defined as

$$Y \longrightarrow S(Y) := \nabla_Y Z\ .$$

In local coordinates, S takes the form

$$S = \nabla_\mu Z^\nu\, dx^\mu \otimes \partial_\nu\ .$$

It follows from the Leibniz rule — or, equivalently, from the definitions in Section 2.3 — that we have

$$\begin{aligned}(\nabla_X S)(Y) &= \nabla_X(S(Y)) - S(\nabla_X Y)\\ &= \nabla_X\nabla_Y Z - \nabla_{\nabla_X Y}Z\ .\end{aligned}$$

The commutator of the derivatives can then be calculated as

$$\begin{aligned}(\nabla_X S)(Y) - (\nabla_Y S)(X) &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{\nabla_X Y}Z + \nabla_{\nabla_Y X}Z\\ &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\\ &\quad +\nabla_{[X,Y]}Z - \nabla_{\nabla_X Y}Z + \nabla_{\nabla_Y X}Z\\ &= R(X,Y)Z - \nabla_{T(X,Y)}Z\ .\end{aligned} \tag{2.5.4}$$

Writing ∇S in the usual form

$$\nabla S = \nabla_\sigma S_\mu{}^\nu\, dx^\sigma \otimes dx^\mu \otimes \partial_\nu = \nabla_\sigma\nabla_\mu Z^\nu\, dx^\sigma \otimes dx^\mu \otimes \partial_\nu\ ,$$

we are thus led to

$$\nabla_\mu\nabla_\nu Z^\alpha - \nabla_\nu\nabla_\mu Z^\alpha = R_{\mu\nu\sigma}{}^\alpha Z^\sigma - T^\sigma{}_{\mu\nu}\nabla_\sigma Z^\alpha\ . \tag{2.5.5}$$

In the important case of vanishing torsion, the coordinate-component equivalent of (2.5.3) is thus

$$\boxed{\nabla_\mu\nabla_\nu X^\alpha - \nabla_\nu\nabla_\mu X^\alpha = R_{\mu\nu\sigma}{}^\alpha X^\sigma}\ . \tag{2.5.6}$$

A calculation identical to that in the proof of Proposition 2.5.1 gives, again for torsionless connections,

$$\nabla_\mu\nabla_\nu a_\alpha - \nabla_\nu\nabla_\mu a_\alpha = -R_{\mu\nu\alpha}{}^\sigma a_\sigma\ . \tag{2.5.7}$$

For a general tensor t and torsion-free connection, each tensor index comes with a corresponding Riemann tensor term:

$$\begin{aligned}\nabla_\mu\nabla_\nu t_{\alpha_1\ldots\alpha_r}{}^{\beta_1\ldots\beta_s}-\nabla_\nu\nabla_\mu t_{\alpha_1\ldots\alpha_r}{}^{\beta_1\ldots\beta_s} &=\\ -R_{\mu\nu\alpha_1}{}^{\sigma}t_{\sigma\ldots\alpha_r}{}^{\beta_1\ldots\beta_s}-\ldots-R_{\mu\nu\alpha_r}{}^{\sigma}t_{\alpha_1\ldots\sigma}{}^{\beta_1\ldots\beta_s}&\\ +R_{\mu\nu\sigma}{}^{\beta_1}t_{\alpha_1\ldots\alpha_r}{}^{\sigma\ldots\beta_s}+\ldots+R_{\mu\nu\sigma}{}^{\beta_s}t_{\alpha_1\ldots\alpha_r}{}^{\beta_1\ldots\sigma}\ .&\end{aligned} \tag{2.5.8}$$

From now on we assume a Levi-Civita connection; note that the convention used in Woodhouse's book [9] on the positioning of the Riemann tensor indices is opposite to the one I have always been using, which is bound to create problems once in a while

One of the fundamental results in Riemannian geometry is:

THEOREM 2.5.3 *There exists a coordinate system in which the metric tensor field has vanishing second derivatives at p if and only if its Riemann tensor vanishes at p. Furthermore, there exists a coordinate system in which the metric tensor field has constant entries near p if and only if the Riemann tensor vanishes near p.*

PROOF: The condition is necessary, since $R_{\alpha\beta\gamma}{}^{\delta}$ is a tensor. The sufficiency will be admitted. □

•2.5.3 In a coordinate basis $\{e_a\}=\{\partial_\mu\}$ we find[2] (recall that $[\partial_\mu,\partial_\nu]=0$)

•2.5.3: this is an alternative derivation of the explicit form of the Riemann, but note that it uses a different convention on the positioning of indices; you could try to rewrite this calculation in the convention used during the lectures

$$\begin{aligned}R^\alpha{}_{\beta\gamma\delta} &:= \langle dx^\alpha, R(\partial_\gamma,\partial_\delta)\partial_\beta\rangle\\ &= \langle dx^\alpha,\nabla_\gamma\nabla_\delta\partial_\beta\rangle-\langle\cdots\rangle_{\delta\leftrightarrow\gamma}\\ &= \langle dx^\alpha,\nabla_\gamma(\Gamma^\sigma{}_{\beta\delta}\partial_\sigma)\rangle-\langle\cdots\rangle_{\delta\leftrightarrow\gamma}\\ &= \langle dx^\alpha,\partial_\gamma(\Gamma^\sigma{}_{\beta\delta})\partial_\sigma+\Gamma^\rho{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta}\partial_\rho\rangle-\langle\cdots\rangle_{\delta\leftrightarrow\gamma}\\ &= \{\partial_\gamma\Gamma^\alpha{}_{\beta\delta}+\Gamma^\alpha{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta}\}-\{\cdots\}_{\delta\leftrightarrow\gamma}\ ,\end{aligned}$$

leading finally to

$$\boxed{\boxed{R^\alpha{}_{\beta\gamma\delta}=\partial_\gamma\Gamma^\alpha{}_{\beta\delta}-\partial_\delta\Gamma^\alpha{}_{\beta\gamma}+\Gamma^\alpha{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta}-\Gamma^\alpha{}_{\sigma\delta}\Gamma^\sigma{}_{\beta\gamma}}}\ . \tag{2.5.9}$$

In a general frame some supplementary commutator terms will appear in the formula for $R^a{}_{bcd}$.

[2]The reader is warned that certain authors use a different sign convention either for $R(X,Y)Z$, or for $R^\alpha{}_{\beta\gamma\delta}$, or both. A useful table that lists the sign conventions for a series of standard GR references can be found on the backside of the front cover of [4].

2.5.1 Symmetries

Here is a full list of algebraic symmetries of the curvature tensor of the Levi-Civita connection:

1. directly from the definition, we obtain
$$R_{\alpha\beta\gamma}{}^{\delta} = -R_{\beta\alpha\gamma}{}^{\delta} \; ; \tag{2.5.10}$$

2. the next symmetry, known as the *first Bianchi identity*, is less obvious:
$$R_{\alpha\beta\gamma}{}^{\delta} + R_{\beta\gamma\alpha}{}^{\delta} + R_{\gamma\alpha\beta}{}^{\delta} = 0 \; ; \tag{2.5.11}$$

3. and finally we have the pair-interchange symmetry:
$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \; . \tag{2.5.12}$$

   Here, of course, $R_{\alpha\beta\gamma\delta} = g_{\delta\sigma}R_{\alpha\beta\gamma}{}^{\sigma}$.

It is a not obvious fact that these are all independent algebraic identities satisfied by $R_{\alpha\beta\gamma\delta}$.

As a consequence of (2.5.10) and (2.5.12) we find

$$R_{\alpha\beta\delta\gamma} = R_{\delta\gamma\alpha\beta} = -R_{\gamma\delta\alpha\beta} = -R_{\alpha\beta\gamma\delta} \; ,$$

and so the Riemann tensor is also anti-symmetric in its last two indices:

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \; . \tag{2.5.13}$$

The Ricci tensor is defined as

$$R_{\alpha\beta} := R_{\alpha\sigma\beta}{}^{\sigma} \; .$$

The pair-interchange symmetry implies that the Ricci tensor is symmetric:

$$R_{\alpha\beta} = g^{\sigma\rho}R_{\alpha\sigma\beta\rho} = g^{\sigma\rho}R_{\beta\rho\alpha\sigma} = R_{\beta\alpha} \; .$$

For proofs, see Woodhouse [9] Section 5.7. Here we will only give a proof of

$$\boxed{R_{abcd} = R_{cdab}} \; . \tag{2.5.14}$$

We suppose that the metric is twice-differentiable. By point 2. of Proposition 2.4.1 in a neighborhood of any point $p \in M$ there exists a coordinate system in which the connection coefficients $\Gamma^\alpha{}_{\beta\gamma}$ vanish at p. Equation (2.5.9) evaluated at p therefore reads

$$\begin{aligned}
R^\alpha{}_{\beta\gamma\delta} &= \partial_\gamma\Gamma^\alpha{}_{\beta\delta} - \partial_\delta\Gamma^\alpha{}_{\beta\gamma} \\
&= \frac{1}{2}\Big\{g^{\alpha\sigma}\partial_\gamma(\partial_\delta g_{\sigma\beta} + \partial_\beta g_{\sigma\delta} - \partial_\sigma g_{\beta\delta}) \\
&\quad -g^{\alpha\sigma}\partial_\delta(\partial_\gamma g_{\sigma\beta} + \partial_\beta g_{\sigma\gamma} - \partial_\sigma g_{\beta\gamma})\Big\} \\
&= \frac{1}{2}g^{\alpha\sigma}\Big\{\partial_\gamma\partial_\beta g_{\sigma\delta} - \partial_\gamma\partial_\sigma g_{\beta\delta} - \partial_\delta\partial_\beta g_{\sigma\gamma} + \partial_\delta\partial_\sigma g_{\beta\gamma}\Big\} .
\end{aligned}$$

Equivalently,

$$R_{\sigma\beta\gamma\delta}(0) = \frac{1}{2}\Big\{\partial_\gamma\partial_\beta g_{\sigma\delta} - \partial_\gamma\partial_\sigma g_{\beta\delta} - \partial_\delta\partial_\beta g_{\sigma\gamma} + \partial_\delta\partial_\sigma g_{\beta\gamma}\Big\}(0) . \quad (2.5.15)$$

The first term goes to the last one under interchange of $\sigma\beta$ with $\gamma\delta$; similarly for the second and the third, and (2.5.14) follows.

Further, the indices $\sigma\beta\gamma$ on the first term in (2.5.15) form a cyclic permutation of those in the second; similarly for the third and the fourth one. This proves the first Bianchi identity (2.5.11).

We finish this section by mentioning a differential identity satisfied by the Riemmann tensor, known as the *second Bianchi identity*; see one of the problem sheets for a derivation:

$$\nabla_\alpha R_{\beta\gamma\delta}{}^\sigma + \nabla_\beta R_{\gamma\alpha\delta}{}^\sigma + \nabla_\gamma R_{\alpha\beta\delta}{}^\sigma = 0 . \quad (2.5.16)$$

Chapter 3

Curved space-time

3.1 Summary of basic ideas

1. "Special relativity holds over small distances and short times in local inertial frames". This is implemented by allowing the Minkowski metric
$$\eta = dt^2 - dx^2 - dy^2 - dz^2$$
to be replaced by a "metric tensor" with Lorentzian signature
$$g = g_{\mu\nu} dx^\mu dx^\nu \ .$$
In particular it follows that physical observers move along curves with $g(\dot\gamma, \dot\gamma) > 0$ - such curves are called *timelike*.

2. Gravity appears as the relative acceleration of nearby local inertial frames.

3. **Principle of general covariance:** the theory should be formulated in a way which does not give any special role to any particular coordinate system.

4. Vacuum Einstein equations: in vacuum, the Ricci tensor $R_{\mu\nu} := R_{\mu\alpha\nu}{}^{\alpha}$ vanishes.

5. Freely falling observers move along timelike geodesics.[1]

6. Space-time is a manifold!

3.1.1 Manifolds

Definition 3.1.1 *An n–dimensional manifold is a set M equipped with the following:* $\bullet$*3.1.1*

$\bullet$*3.1.1: this is not examinable*

[1]There is actually a sense in which this follows from point 4, but for the purposes of this course we will admit this as an axiom.

1. *topology: a "connected Hausdorff paracompact topological space" (think of nicely looking subsets of* $\mathbb{R}^{1+n}$*, like spheres, hyperboloids, and such), together with*

2. *local charts: a collection of coordinate patches* $(\mathscr{U}, x^i)$ *covering* M*, where* $\mathscr{U}$ *is an open subset of* M*, with the functions* $x^i : \mathscr{U} \to \mathbb{R}^n$ *being continuous. One further requires that the maps*

$$M \supset \mathscr{U} \ni p \mapsto (x^1(p), \ldots, x^n(p)) \in \mathscr{V} \subset \mathbb{R}^n$$

are homeomorphisms.

3. *compatibility: given two overlapping coordinate patches,* $(\mathscr{U}, x^i)$ *and* $(\widetilde{\mathscr{U}}, \tilde{x}^i)$*, with corresponding sets* $\mathscr{V}, \widetilde{\mathscr{V}} \subset \mathbb{R}^n$*, the maps* $\tilde{x}^j \mapsto x^i(\tilde{x}^j)$ *are smooth diffeomorphisms wherever defined: this means that they are bijections differentiable as many times as one wishes, with*

$$\det \left[\frac{\partial x^i}{\partial \tilde{x}^j} \right] \textit{ nowhere vanishing}\,.$$

Definition of differentiability: A function on M *is smooth if it is smooth when expressed in terms of local coordinates. Similarly for tensors.*

Examples:

1. $\mathbb{R}^n$ with the usual topology, one single global coordinate patch.

2. A sphere: use stereographic projection to obtain two overlapping coordinate systems (or use spherical angles, but then one must avoid borderline angles, so they don't cover the whole manifold!).

3. We will use several coordinate patches (in fact, five), to describe the Schwarzschild black hole, though one spherical coordinate system would suffice.

4. Let $f : \mathbb{R}^n \to \mathbb{R}$, and define $N := f^{-1}(0)$. If ∇f has no zeros on N, then N is a smooth $(n-1)$–dimensional manifold. This construction leads to a plethora of examples. For example, if $f = \sqrt{(x^1)^2 + \ldots + (x^n)^2} - R$, with $R > 0$, then N is a sphere of radius R.

3.1.2 Geodesic deviation (Jacobi equation), tidal forces

What does this theory have to do with gravitation? To understand this, let us look at families of geodesics.

Suppose that we have a one parameter family of geodesics

$$\gamma(s, \lambda) \text{ (in local coordinates, } (\gamma^\alpha(s, \lambda))),$$

where s is the parameter along the geodesic, and λ is a parameter which distinguishes the geodesics. Set

$$Z(s,\lambda) := \frac{\partial\gamma(s,\lambda)}{\partial\lambda} \equiv \frac{\partial\gamma^\alpha(s,\lambda)}{\partial\lambda}\partial_\alpha \ ;$$

for each λ this defines a vector field Z along $\gamma(s,\lambda)$, which measures how nearby geodesics deviate from each other, since, to first order, using a Taylor expansion,

$$\gamma^\alpha(s,\lambda) = \gamma^\alpha(s,\lambda_0) + Z^\alpha(\lambda-\lambda_0) + O((\lambda-\lambda_0)^2) \ .$$

To measure how a vector field W changes along $s \mapsto \gamma(s,\lambda)$, one introduces the differential operator D/ds, defined as

$$\begin{aligned} \frac{DW^\mu}{ds} &:= \frac{\partial(W^\mu\circ\gamma)}{\partial s} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta W^\alpha & (3.1.1)\\ &= \dot\gamma^\beta\frac{\partial W^\mu}{\partial x^\beta} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta W^\alpha & (3.1.2)\\ &= \dot\gamma^\beta\nabla_\beta W^\mu \ . & (3.1.3)\end{aligned}$$

(It would perhaps be more logical to write $\frac{DW^\mu}{\partial s}$ in the current context, but people never do that.) The last two lines only make sense if W is defined in a whole neighbourhood of γ, but for the first it suffices that $W(s)$ be defined along $s \mapsto \gamma(s,\lambda)$. (One possible way of making sense of the last two lines is to extend W^μ to any smooth vector field defined in a neighorhood of $\gamma^\mu(s,\lambda)$, and note that the result is independent of the particular choice of extension because the equation involves only derivatives tangential to $s \mapsto \gamma^\mu(s,\lambda)$.)

Analogously one sets

$$\begin{aligned} \frac{DW^\mu}{d\lambda} &:= \frac{\partial(W^\mu\circ\gamma)}{\partial\lambda} + \Gamma^\mu{}_{\alpha\beta}\partial_\lambda\gamma^\beta W^\alpha & (3.1.4)\\ &= \partial_\lambda\gamma^\beta\frac{\partial W^\mu}{\partial x^\beta} + \Gamma^\mu{}_{\alpha\beta}\partial_\lambda\gamma^\beta W^\alpha & (3.1.5)\\ &= Z^\beta\nabla_\beta W^\mu \ . & (3.1.6)\end{aligned}$$

Note that since $s \to \gamma(s,\lambda)$ is a geodesic we have from (3.1.1) and (3.1.3)

$$\frac{D^2\gamma^\mu}{ds^2} := \frac{D\dot\gamma^\mu}{ds} = \frac{\partial^2\gamma^\mu}{\partial s^2} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta\dot\gamma^\alpha = 0 \ . \qquad (3.1.7)$$

(This is sometimes written as $\dot\gamma^\alpha\nabla_\alpha\dot\gamma^\mu = 0$, which is again an abuse of notation since typically we will only know $\dot\gamma^\mu$ as a function of s, and so there is no such thing as $\nabla_\alpha\dot\gamma^\mu$.) Furthermore,

$$\frac{DZ^\mu}{ds} \underbrace{=}_{(3.1.1)} \frac{\partial^2\gamma^\mu}{\partial s\partial\lambda} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta\partial_\lambda\gamma^\alpha \underbrace{=}_{(3.1.4)} \frac{D\dot\gamma^\mu}{d\lambda} \ , \qquad (3.1.8)$$

(The abuse-of-notation derivation of the same formula proceeds as:

$$\nabla_{\dot\gamma} Z^\mu = \dot\gamma^\nu \nabla_\nu Z^\mu = \dot\gamma^\nu \nabla_\nu \partial_\lambda \gamma^\mu \underbrace{=}_{(3.1.3)} \frac{\partial^2 \gamma^\mu}{\partial s \partial \lambda} + \Gamma^\mu{}_{\alpha\beta} \dot\gamma^\beta \partial_\lambda \gamma^\alpha \underbrace{=}_{(3.1.6)} Z^\beta \nabla_\beta \dot\gamma^\mu = \nabla_Z \dot\gamma^\mu \ , \tag{3.1.9}$$

which can then be written as

$$\nabla_{\dot\gamma} Z = \nabla_Z \dot\gamma \ .) \tag{3.1.10}$$

One can now repeat the calculation leading to (2.5.1) to obtain, for any vector field W defined along $\gamma^\mu(s,\lambda)$,

$$\frac{D}{ds}\frac{D}{d\lambda} W^\mu - \frac{D}{d\lambda}\frac{D}{ds} W^\mu = R_{\alpha\beta\delta}{}^\mu \dot\gamma^\alpha Z^\beta W^\delta \ . \tag{3.1.11}$$

If $W^\mu = \dot\gamma^\mu$ the second term at the left-hand-side is zero, and from $\frac{D}{d\lambda}\dot\gamma = \frac{D}{ds} Z$ we obtain

$$\frac{D^2 Z^\mu}{ds^2}(s) = R_{\alpha\beta\sigma}{}^\mu \dot\gamma^\alpha Z^\beta \dot\gamma^\sigma \ . \tag{3.1.12}$$

We have obtained an equation known as the *Jacobi equation*, or as the *geodesic deviation equation*; in index-free notation:

$$\boxed{\frac{D^2 Z}{ds^2} = R(\dot\gamma, Z)\dot\gamma} \ . \tag{3.1.13}$$

Solutions of (3.1.13) are called *Jacobi fields* along γ.

Equation (3.1.13) shows that curvature causes relative accelerations between neighboring geodesics. Keeping in mind that gravitational force and acceleration are indistinguishable, we say that curvature produces a "gravitational tidal force" between freely falling nearby observers.

The advantage of the abuse-of-notation equations above, which can be justified by the extension-artifact already mentioned, is that one can invoke the result of Proposition 2.5.1, instead of repeating its calculations, to obtain (3.1.11):

$$\begin{aligned}
\frac{D^2 Z^\mu}{ds^2}(s) &= \dot\gamma^\alpha \nabla_\alpha (\dot\gamma^\beta \nabla_\beta Z^\mu) \\
&= \dot\gamma^\alpha \nabla_\alpha (Z^\beta \nabla_\beta \dot\gamma^\mu) \\
&= (\dot\gamma^\alpha \nabla_\alpha Z^\beta) \nabla_\beta \dot\gamma^\mu + Z^\beta \dot\gamma^\alpha \nabla_\alpha \nabla_\beta \dot\gamma^\mu \\
&= (\dot\gamma^\alpha \nabla_\alpha Z^\beta) \nabla_\beta \dot\gamma^\mu + Z^\beta \dot\gamma^\alpha (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) \dot\gamma^\mu + Z^\beta \dot\gamma^\alpha \nabla_\beta \nabla_\alpha \dot\gamma^\mu \\
&= (\dot\gamma^\alpha \nabla_\alpha Z^\beta) \nabla_\beta \dot\gamma^\mu + Z^\beta \dot\gamma^\alpha R_{\alpha\beta\sigma}{}^\mu \dot\gamma^\sigma + Z^\beta \dot\gamma^\alpha \nabla_\beta \nabla_\alpha \dot\gamma^\mu \\
&= (\dot\gamma^\alpha \nabla_\alpha Z^\beta) \nabla_\beta \dot\gamma^\mu + Z^\beta \dot\gamma^\alpha R_{\alpha\beta\sigma}{}^\mu \dot\gamma^\sigma + Z^\beta \nabla_\beta \underbrace{(\dot\gamma^\alpha \nabla_\alpha \dot\gamma^\mu)}_{0} - (Z^\beta \nabla_\beta \dot\gamma^\alpha) \nabla_\alpha \dot\gamma^\mu \ .
\end{aligned}$$

A renaming of indices in the first and the last term gives

$$(\dot\gamma^\alpha \nabla_\alpha Z^\beta) \nabla_\beta \dot\gamma^\mu - (Z^\beta \nabla_\beta \dot\gamma^\alpha) \nabla_\alpha \dot\gamma^\mu = (\dot\gamma^\alpha \nabla_\alpha Z^\beta - Z^\alpha \nabla_\alpha \dot\gamma^\beta) \nabla_\beta \dot\gamma^\mu \ ,$$

which is zero by (3.1.10). This leads again to (3.1.12).

3.2 Einstein equations and matter

We have already seen in lectures that Einstein's equations in vacuum read

$$R_{\mu\nu} = 0$$

where $R_{\mu\nu}$ is the Ricci tensor. Anticipating, in the presence of matter the right-hand-side will not be zero, but will involve an object describing the density of energy of matter fields.

The idea is: energy produces curvature. So we need a tensor with two indices which will describe the energy contents of the matter fields.

This requires examining the matter models. We start with the simplest one, that of dust.

3.2.1 Dust in special and general relativity

By definition, *dust* is a cloud of non-interacting particles, whose velocities vary smoothly from point to point in space-time.

So at each point we have a scalar ρ which represents the density of the dust: this is the rest mass per unit volume measured in a frame in which the particles are at rest. For example, if there are n particles per unit volume and each has mass m, then $\rho = nm$.

We wish to calculate the energy density of the dust in moving frames.

By definition, a rest frame is a frame in which the particles do not move, so that their space velocity is zero, and therefore their velocity four-vector is

$$u = u^\mu \partial_\mu = \partial_t \Longleftrightarrow (u^\mu) = (1, \vec{0}) \ .$$

Let an observer move with velocity $\vec{v}$ with respect to the dust, so she has a four-velocity vector

$$(v^\mu) = (\gamma = \frac{1}{\sqrt{1-\vec{v}^2}}, \vec{v}) \ .$$

Choosing a coordinate system so that the velocity is aligned along the x axis and pointing in the positive direction, the observer has velocity v along the x-axis.

Let there be n particles of rest mass m in a box with sides dx, dy and dz in the reference frame of the dust.

The observer sees n particles of rest mass m in a box of size $1/\gamma dx$ (contraction factor!), dy and dz, with velocity $-\vec{v}$, and therefore energy

$$mn\gamma$$

in a volume

$$dx\,dy\,dz/\gamma\ ,$$

and hence a density of

$$mn\gamma^2 = \rho(u_\mu v^\mu)^2 = \underbrace{\rho u_\mu u_\nu}_{=:T_{\mu\nu}} v^\mu v^\nu\ . \tag{3.2.1}$$

The tensor

$$T_{\mu\nu} = \rho u_\mu u_\nu$$

is called the *energy-momentum tensor of dust*, with energy density ρ, and is used to measure the energy density of dust in moving frames, in a sense made clear by equation (3.2.1).

Energy-momentum tensors belong to the class of basic objects in general relativity, as they provide the source-part of the Einstein equations.

Another example of energy-momentum tensor is given by the Maxwell energy-momentum tensor

$$T_{\mu\nu} = \epsilon_0 \left(F_{\mu\alpha} F^\alpha{}_\nu + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right)\ . \tag{3.2.2}$$

see Section 3.3 of Woodhouse's book.
•3.2.1

•3.2.1: *in the general relativity literature one often uses units in which $\epsilon_0 = 1/4\pi$*

3.2.2 The continuity equation

An important property of energy-momentum tensors *IN SPECIAL RELATIVITY* is that they satisfy a *continuity equation*:

$$\partial_\nu T^{\mu\nu} = 0\ . \tag{3.2.3}$$

As an exercice, you can check that the divergence identity (3.2.3) for the Maxwell energy-momentum tensor (3.2.2) follows from the Maxwell equations.

In order to verify (3.2.3) for dust, we need first to know what the equations are. Since we assume that the particles are non-interacting, each of them moves along a straight line. Now, straight lines are geodesics in Minkowski space-time, so if u^μ is the vector tangent to each geodesic followed by the particles, we have seen in (3.1.7) that the geodesic equation can be written as

$$u^\mu \nabla_\mu u^\nu = 0\ . \tag{3.2.4}$$

Since the number of particles is conserved we also have the *conservation equation*

$$\nabla_\mu(\rho u^\mu) = 0\ . \tag{3.2.5}$$

These are thus the equations for our model.

These are local equations, and one uses the correspondence principle to carry them over to general relativity, where the covariant derivative becomes the covariant derivative of a possibly curved metric.

Whether in curved space-time or not, if we calculate the divergence of the energy-momentum tensor we obtain

$$\nabla_\mu(\rho u^\mu u^\nu) = \nabla_\mu(\rho u^\mu)u^\nu + \rho u^\mu \nabla_\mu u^\nu = 0 \,, \tag{3.2.6}$$

where the first term is zero by the continuity equation (3.2.5), and the second vanishes by the geodesic equation (3.2.4).

In fact, this equation is equivalent to (3.2.4)-(3.2.5) in regions where ρ does not vanish if we remember the condition that

$$u_\nu u^\nu = 1 \,. \tag{3.2.7}$$

Indeed, by (3.2.7) we have

$$0 = \nabla_\mu(u_\nu u^\nu) = \nabla_\mu(g_{\alpha\nu}u^\alpha u^\nu) = 2g_{\alpha\nu}u^\alpha \nabla_\mu u^\nu = 2u_\nu \nabla_\mu u^\nu \,.$$

If we multiply (3.2.6) with u^ν and use the last equation we recover the continuity equation, but then the geodesic character of u^μ follows.

In special relativity it is again true that the divergence of the Maxwell energy-momentum tensor (3.2.2) vanishes when the source-free Maxwell equations hold.

3.2.3 Einstein equations with sources

The energy-momentum tensor $T_{\mu\nu}$ provides a good candidate for the source term in Einstein's theory of gravitation. Keeping in mind the special relativity correspondence principle, and our analysis of dust and of the Maxwell field, the energy-momentum of matter fields will thus be described by a symmetric tensor satisfying

$$\nabla_\mu T^{\mu\nu} = 0 \,, \tag{3.2.8}$$

or, equivalently,

$$\nabla^\mu T_{\mu\nu} = 0 \,. \tag{3.2.9}$$

We thus need to write an equation which is compatible with this restriction. For this, we note the important identity:

$$\nabla^\mu(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) = 0 \,, \tag{3.2.10}$$

where

$$R := R^\alpha{}_\alpha = R^{\alpha\beta}{}_{\alpha\beta} \,.$$

To prove (3.2.10), recall the second Bianchi identity

$$\nabla_\mu R_{\nu\rho\alpha\beta} + \nabla_\nu R_{\rho\mu\alpha\beta} + \nabla_\rho R_{\mu\nu\alpha\beta} = 0 \ . \tag{3.2.11}$$

Multiplying by $g^{\mu\alpha}g^{\nu\beta}$ we obtain

$$\nabla^\alpha R^\beta{}_{\rho\alpha\beta} + \nabla^\beta R_\rho{}^\alpha{}_{\alpha\beta} + \nabla_\rho R^{\alpha\beta}{}_{\alpha\beta} = 0 \ , \tag{3.2.12}$$

which is another way of writing (3.2.10).

Recalling that for any constant Λ we have

$$\nabla^\mu(\Lambda g_{\mu\nu}) = 0 \ ,$$

we are led to an equation which will be compatible with (3.2.9):

$$\boxed{R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \ .} \tag{3.2.13}$$

The constant κ can be determined by considering the Newtonian limit, where $g_{\mu\nu}$ is very close to the Minkowski metric, and all velocities are very small compared with the speed of light, leading to

$$\kappa = -\frac{8\pi G}{c^4} \ .$$

The constant Λ is called *the cosmological constant*, and current state-of-the art observations [6, 8] indicate strongly that Λ is nonzero. Indeed, the current standard model of cosmology requires a cosmological constant which, from Hubble observations, is measured to be on the order of $10^{-35}s^{-2}$, very small but non-zero. This makes a difference for cosmology, but not for describing black holes, or the solar system. So from now on we will assume that $\Lambda = 0$, and use units $G = c = 1$, so that (3.2.13) becomes

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -8\pi T_{\mu\nu} \ . \tag{3.2.14}$$

Chapter 4

The Schwarzschild metric

Chapter 5

The Schwarzschild metric

5.1 The metric

The simplest stationary solutions describing compact isolated objects are the spherically symmetric ones. The flagship example is the *Schwarzschild metric*:

$$g = (1 - \tfrac{2m}{r})dt^2 - \tfrac{dr^2}{1-\frac{2m}{r}} - r^2 d\Omega^2 \ , \qquad (5.1.1)$$

$$t \in \mathbb{R} \ , \ r \neq 2m, 0 \ . \qquad (5.1.2)$$

Here $d\Omega^2$ denotes the metric of the round unit 2-sphere,

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2 \ .$$

A theorem due to Birkhoff shows that:

Theorem 5.1.1 *In a vacuum region, away from the set $\{r = 2m\}$ any spherically symmetric metric can* locally *be written in the Schwarzschild form, for some mass parameter m.*

•5.1.1

•5.1.1: see Woodhouse or Besse [1] for two very different verifications that this metric satisfies the vacuum Einstein equations

We conclude that the hypothesis of spherical symmetry implies in vacuum, at least locally, the existence of two further symmetries: translations in t and t–reflections $t \to -t$. More precisely, we obtain *time translations* and *time-reflections* in the region where $1 - 2m/r > 0$ (a metric with those two properties is called *static*). However, in the region where $r < 2m$ the notation "t" for the coordinate appearing in (5.1.1) is misleading, as t is then a space-coordinate, and r is a time one. So in this region t–translations are actually translations in space.

The above requires some comments and definitions:

Time functions

First, we need to define the notion of *time orientation*. This is the decision about which timelike vectors are future-pointing, and which ones are past-pointing. In special relativity this is taken for granted: in coordinates where the Minkowski metric η takes the form

$$\eta = dt^2 - dx^2 - dy^2 - dz^2 \;, \tag{5.1.3}$$

a timelike vector $X^\mu\partial_\mu$ is said to be future pointing if $X^0 > 0$. But, it should be realized that this is a question of conventions: we could very well agree that future-pointing vectors are those with negative X^0. We will shortly meet a situation where such a decision will have to be made.

A function f will be called a *time function* if ∇f is everywhere timelike future pointing. A coordinate, say y^0 will be said to be a *time coordinate* if y^0 is a time function.

So, for example, $f = t$ on Minkowski space-time is a time function: indeed, in canonical coordinates as in (5.1.3)

$$\nabla t = \eta^{\mu\nu}\partial_\mu t\,\partial_\nu = \eta^{0\nu}\partial_\nu = \partial_t \;,$$

and so

$$\eta(\nabla t, \nabla t) = \eta(\partial_t, \partial_t) = 1 \;.$$

On the other hand, consider $f = t$ in the Schwarzschild metric: the inverse metric now reads

$$g^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{1-\frac{2m}{r}}\partial_t^2 - (1-\frac{2m}{r})\partial_r^2 - r^{-2}(\partial_\theta^2 + \sin^{-2}\theta\partial_\varphi^2) \;, \tag{5.1.4}$$

and so

$$\nabla t = g^{\mu\nu}\partial_\mu t\,\partial_\nu = g^{0\nu}\partial_\nu = \frac{1}{1-\frac{2m}{r}}\partial_t \;.$$

The length-squared of ∇t is thus

$$g(\nabla t, \nabla t) = \frac{g(\partial_t, \partial_t)}{(1-\frac{2m}{r})^2} = \frac{1}{1-\frac{2m}{r}} \begin{cases} >0, & r > 2m; \\ <0, & r < 2m. \end{cases} \;.$$

We conclude that t *is a time function in the region* $\{r > 2m\}$, but is *not* on the manifold $\{r < 2m\}$.

A similar calculation for ∇r gives

$$g(\nabla r, \nabla r) = (1-\frac{2m}{r})^2 g(\partial_r, \partial_r) = -(1-\frac{2m}{r}) \begin{cases} <0, & r > 2m; \\ >0, & r < 2m. \end{cases} \;.$$

So r is a *time function* in the region $\{r < 2m\}$, *if* the time orientation is chosen so that r is increasing towards the future. On the other hand, the alternative choice of time-orientation implies that *minus* r is a time function in this region.

$r = 0$

Throughout this chapter we will assume

$$m > 0 \,,$$

because $m < 0$ leads to metrics which contain "naked singularities", in the following sense: for $m < 0$, on each space-like surface $\{t = \text{const}\}$ the set $\{r = 0\}$ can be reached along curves of finite length. •5.1.2 But we have (see, *e.g.*, `http://grtensor.phy.queensu.ca/NewDemo`)

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{48m^2}{r^6} \,, \tag{5.1.5}$$

•*5.1.2: you are not expected to justify this fact, nor to calculate the expression (5.1.5), but you are expected to be aware of the existence of a curvature scalar which diverges as $r = 0$ is approached.*

which shows that the geometry is singular at $r = 0$, whatever $m \in \mathbb{R}^*$.

The advantage of $m > 0$ is the occurrence, as will be seen shortly, of the event horizon $\{r = 2m\}$: the singular set $\{r = 0\}$ is then "hidden" behind an event horizon, which is considered to be less unpleasant than the situation with $m < 0$, where no such horizon occur.

Eddington-Finkelstein extension

The metric (5.1.1) is singular as $r = 2m$ is approached. It turns out that this singularity is related to a poor choice of coordinates (one talks about "a coordinate singularity"); the simplest way to see it is to replace t by a new coordinate v, which will be chosen to cancel out the singularity in g_{rr}: if we set

$$v = t + f(r) \,,$$

we find $dv = dt + f'dr$, so that

$$\begin{aligned} (1 - \frac{2m}{r})dt^2 &= (1 - \frac{2m}{r})(dv - f'dr)^2 \\ &= (1 - \frac{2m}{r})(dv^2 - 2f'dv\,dr - (f')^2dr^2) \,. \end{aligned}$$

The offending g_{rr} terms in (5.1.1) will go away if we choose f to satisfy

$$(1 - \frac{2m}{r})(f')^2 = \frac{2}{1 - \frac{2m}{r}} \,.$$

There are two possibilities for the sign; we choose

$$f' = \frac{1}{1 - \frac{2m}{r}} = \frac{r}{r - 2m} = \frac{r - 2m + 2m}{r - 2m} = 1 + \frac{2m}{r - 2m} \,, \tag{5.1.6}$$

leading to

$$v = t + r + 2m\ln(r - 2m) \,.$$

The alternative choice amounts to introducing another coordinate

$$u = t - f(r) \ , \tag{5.1.7}$$

with f still as in (5.1.6), we will return to this possibility shortly.

This brings g to the form

$$g = \left(1 - \frac{2m}{r}\right) dv^2 - 2dv\, dr - r^2 d\Omega^2 \ . \tag{5.1.8}$$

We have $\det g = -r^4 \sin^2\theta$, with all coefficients of g smooth, which shows that g is a well defined Lorentzian metric on the set

$$\{v \in \mathbb{R} \ , r \in (0,\infty)\} \times S^2 \ . \tag{5.1.9}$$

More precisely, (5.1.8)-(5.1.9) defines an analytic extension of the original space-time (5.1.1).

The coordinates (v, r, θ, φ) are called *"retarded Eddington-Finkelstein coordinates"*.

For further reference, note that increasing t corresponds to increasing v.

We claim:

THEOREM 5.1.2 *The region $\{r \leq 2m\}$ for the metric (5.1.8) is a* black hole region*, in the sense that*

observers, or signals, can enter this region, but can never leave it. (5.1.10)

PROOF: We have already seen that either r or *minus* r is a time function on the region $\{r < 2m\}$. Now, recall that observers in general relativity always move on *future directed timelike curves*, that is, curves with timelike future directed tangent vector. But time functions are strictly monotonous along future directed causal curves: indeed, let $\gamma(s)$ be such a curve, and let f be a time function, then

$$\frac{d(f \circ \gamma)}{ds} = \dot\gamma^\mu \partial_\mu f = \dot\gamma^\mu g_{\mu\nu} g^{\sigma\nu} \partial_\sigma f = g_{\mu\nu} \dot\gamma^\mu \nabla^\nu f \ .$$

Since both $\dot\gamma$ and ∇f are timelike future directed, their scalar product is positive, as desired.

It follows that, along a future directed causal curve, either r or $-r$ is strictly increasing in the region $\{r < 2m\}$.

Suppose that there exists at least one future directed causal curve γ_0 which enters from $r > 2m$ to $r < 2m$. Then $\dot r$ must have been decreasing somewhere along γ_0 in the region $\{r < 2m\}$. This implies that *the time orientation has to be chosen so that $-r$ is a time function.* But then r is decreasing along every causal future directed γ, So no such curve passing through $\{r < 2m\}$ can cross $\{r = 2m\}$ again, when followed to the future.

To finish the proof, it remains to exhibit one γ_0 which enters $\{r < 2m\}$ from the region $\{r > 2m\}$. For this, consider the radial curve

$$\gamma_0(s) = (v(s), r(s), \theta(s), \varphi(s)) = (s, 3m - s, \frac{\pi}{2}, 0) .$$

Then $\dot\gamma_0 = \partial_v - \partial_r$, hence

$$g(\dot\gamma_0, \dot\gamma_0) = (1 - \frac{2m}{r}) + 2 .$$

We see that γ_0 starts at $r = 3m$ at $s = 0$, and is timelike before, and for some non-empty interval of the parameter s after crossing $\{r = 2m\}$. Next, we have

$$\partial_r|_{v=\text{const}} = \partial_r|_{t=\text{const}} - f'\partial_t ,$$

hence

$$\partial_v = \partial_t , \quad \partial_v - \partial_r|_{v=\text{const}} = (1 + f')\partial_t - \partial_r|_{t=\text{const}} ,$$

which has positive ∂_t–component in the region $\{r > 2m\}$, which concludes the proof.

An alternative proof proceeds as follows: Let $\gamma(s) = (v(s), r(s), \theta(s), \varphi(s))$ be a future directed timelike curve; for the metric (5.1.8) the condition $g(\dot\gamma, \dot\gamma) > 0$ reads

$$-(1 - \frac{2m}{r})\dot v^2 + 2\dot v\dot r + r^2(\dot\theta^2 + \sin^2\theta\dot\varphi^2) < 0 .$$

This implies

$$\dot v\Big(- (1 - \frac{2m}{r})\dot v + 2\dot r\Big) < 0 .$$

It follows that $\dot v$ does not change sign on a timelike curve. As already pointed out, the standard choice of time orientation in the exterior region corresponds to $\dot v > 0$ on future directed curves, so $\dot v$ has to be positive everywhere, which leads to

$$-(1 - \frac{2m}{r})\dot v + 2\dot r < 0 .$$

For $r \le 2m$ the first term is non-negative, which enforces $\dot r < 0$ on all future directed timelike curves in that region. Thus, r is a strictly decreasing function along such curves, which implies that future directed timelike curves can cross the hypersurface $\{r = 2m\}$ only if coming from the region $\{r > 2m\}$. The same conclusion applies for future directed causal curves: it suffices to approximate a causal curve by a sequence of future directed timelike ones.

$\square$

The last theorem motivates the name *black hole event horizon* for $\{r = 2m, v \in \mathbb{R}\} \times S^2$.

The analogous construction using the coordinate u instead of v leads to a *white hole* space-time, with $\{r = 2m\}$ being a *white hole event horizon.* The latter can only be crossed by those future directed causal curves which originate in the region $\{r < 2m\}$. In either case, $\{r = 2m\}$ is a causal membrane which prevents future directed causal curves to go *back and forth.* This will become clearer in Section 5.5.

From (5.1.8) one easily finds the inverse metric:

$$g^{\mu\nu}\partial_\mu\partial_\nu = -2\partial_v\partial_r - (1-\frac{2m}{r})\partial_r^2 - r^{-2}\partial_\theta^2 - r^{-2}\sin^{-2}\theta\partial_\varphi^2 \ . \tag{5.1.11}$$

In particular

$$0 = g^{vv} = g(\nabla v, \nabla v) \ ,$$

which implies that the integral curves of

$$\nabla v = -\partial_r$$

are null, affinely parameterised geodesics: Indeed, let $X = \nabla v$, then

$$X^\alpha\nabla_\alpha X^\beta = \nabla^\alpha v\nabla_\alpha\nabla^\beta v = \nabla^\alpha v\nabla^\beta\nabla_\alpha v = \frac{1}{2}\nabla^\beta(\nabla^\alpha v\nabla_\alpha v) = 0 \ . \tag{5.1.12}$$

So if γ is an integral curve of X (by definition, this means that

$$\dot\gamma^\mu = X^\mu) \ , \tag{5.1.13}$$

we obtain the geodesic equation:

$$X^\alpha\nabla_\alpha X^\beta = \dot\gamma^\alpha\nabla_\alpha\dot\gamma^\beta = 0 \ . \tag{5.1.14}$$

We also have

$$g(\nabla r, \nabla r) = g^{rr} = 1 - \frac{2m}{r} \ , \tag{5.1.15}$$

and since this vanishes at $r = 2m$ we say that the surface $r = 2m$ is *null.* It is reached by all the radial null geodesics $v = \text{const}$, $\theta = \text{const}'$, $\varphi = \text{const}''$, in finite affine time.

The calculation leading to (5.1.14) generalizes to functions f such that ∇f satisfies an equation of the form

$$g(\nabla f, \nabla f) = \psi(f) \ , \tag{5.1.16}$$

for some function ψ; note that $f = r$ satisfies this, in view of (5.1.15). Repeating the calculation done in (5.1.12) with $X = \nabla f$ we instead recover

$$X^\alpha\nabla_\alpha X^\beta = \frac{1}{2}\nabla^\beta(\nabla^\alpha f\nabla_\alpha f) = \frac{1}{2}\psi'\nabla^\beta f = \frac{1}{2}\psi' X^\beta \ . \tag{5.1.17}$$

So if γ satisfies (5.1.13) we obtain

$$\dot\gamma^\alpha\nabla_\alpha\dot\gamma^\beta = \frac{1}{2}\psi'\dot\gamma^\beta \ . \tag{5.1.18}$$

This is again a geodesic, except that now the parameter along γ that results from the defining equation (5.1.13) is not affine; however, a suitable reparameterization of γ will lead to the usual affinely parameterized form of the geodesic equation, as in the second equality of (5.1.14).

5.2 Stationary observers

An observer is called stationary if her coordinates (r, θ, φ) are fixed, so that she is described by a world line

$$t \mapsto \gamma(t) = (t, r, \theta, \varphi) . \tag{5.2.1}$$

The tangent is $\dot\gamma = \partial_t$, is timelike in the region $r > 2m$, and the four velocity there is

$$u := \frac{1}{\sqrt{g(\dot\gamma, \dot\gamma)}}\dot\gamma = \frac{1}{\sqrt{1-\frac{2m}{r}}}\partial_t . \tag{5.2.2}$$

Clearly there is a problem for $r \le 2m$, since the square-root is purely imaginary or vanishes: *there are no stationary observers on or under the horizon $r = 2m$.* Equivalently, the paths defined by (5.2.1) are *not* timelike for $r \le 2m$.

The acceleration four-vector a^μ is defined as

$$a^\mu := \frac{Du^\mu}{ds} = u^\nu \nabla_\mu u^\mu = \frac{1}{\sqrt{1-\frac{2m}{r}}}\nabla_0 u^\mu = \frac{1}{1-\frac{2m}{r}}\Gamma^\mu{}_{00} .$$

We need to calculate the relevant Christoffel symbols

$$\Gamma^\mu{}_{00} = \frac{1}{2}g^{\mu\nu}(2\partial_0 g_{\nu 0} - \partial_\nu g_{00}) = -\frac{1}{2}g^{\mu r}\partial_r g_{00} = \delta^\mu_r \frac{1}{2}(1-\frac{2m}{r})\partial_r(1-\frac{2m}{r}) = \delta^\mu_r(1-\frac{2m}{r})\frac{m}{r^2} .$$

Thus, the four-acceleration vector $a := a^\mu \partial_\mu$ takes the form

$$\frac{m}{r^2}\partial_r ,$$

which looks like the Newtonian force when the parameter m is identified with the mass of the object producing the gravitational field.

It might be preferable to think of the length

$$\sqrt{|g(a,a)|} = \sqrt{|g_{rr}|}\frac{m}{r^2} = \frac{1}{\sqrt{1-\frac{2m}{r}}}\frac{m}{r^2} ,$$

as the correct invariant definition of the strength of the gravitational acceleration. This has Newtonian behavior for large distances, but diverges as the horizon $\{r = 2m\}$ is approached.

An alternative justification of the interpretation of the parameter m as the Newtonian mass will be done in Section 5.4, see (5.4.15).

5.3 The Flamm paraboloid

In this section we try to understand the geometry of the slices with $t=$ const. One way of doing this is to try to embed those slices in four-dimensional Euclidean space. It is not clear that this can be done in general (and, in fact, there are no necessary and sufficient conditions known for this in general; here it works because of spherical symmetry.

We again write the Schwarzschild metric in dimension $n+1$,

$$g_m = -\left(1-\frac{2m}{r^{n-2}}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r^{n-2}}} + r^2 d\Omega^2 \,, \qquad (5.3.1)$$

where, as usual, $d\Omega^2$ is the round unit metric on S^{n-1}. If we set •5.3.1

•5.3.1: *this section is done in all space dimensions n, but in this course we are only interested in n = 3, so you are welcome to ignore the other cases*

$$\mathring{g} = dz^2 + (dx^1)^2 + \ldots + (dx^n)^2 = dz^2 + dr^2 + r^2 d\Omega^2 \,,$$

the metric h induced by $\mathring{g}$ on the the surface $z = z(r)$ reads

$$h = \left(\left(\frac{dz}{dr}\right)^2 + 1\right)dr^2 + r^2 d\Omega^2 \,.$$

This will coincide with the space part of (5.3.1) if we require that

$$\frac{dz}{dr} = \pm\sqrt{\frac{2m}{r^{n-2}-2m}} \,.$$

The equation can be explicitly integrated in dimensions $n=3$ and 4 in terms of elementary functions, leading to

$$z = z_0 \pm \sqrt{2m} \times \left\{ \begin{array}{ll} 2\sqrt{r-2m}, & r > 2m,\ n=3, \\ \ln(r+\sqrt{r^2-2m}), & r > \sqrt{2m},\ n=4. \end{array} \right. \,. \qquad (5.3.2)$$

Solving for $r(z)$, a convenient choice of z_0 leads to

$$r = \left\{ \begin{array}{ll} 2m + z^2/8m, & n=3, \\ \sqrt{2m}\cosh(z/\sqrt{2m}), & n=4. \end{array} \right.$$

In dimension $n=3$ one obtains a paraboloid, as first noted by Flamm. The embeddings are visualized in Figures 5.1 and 5.2.

The qualitative behavior in dimensions $n \geq 5$ is somewhat different, as then $z(r)$ asymptotes to a finite value as r tends to infinity. The embeddings in $n=5$ are visualized in Figure 5.3; in that dimension $z(r)$ can be expressed in terms of elliptic functions, but the final formula is not very illuminating.

We are led to the (momentarily) perplexing observation, that the manifolds $t=$ const, $r > m$, can be "doubled" by attaching another copy of the manifold to itself across the boundary $\{r=2m\}$. This is called *the Einstein-Rosen bridge*.

The positive sign in (5.3.2) corresponds to our usual black hole exterior, while the negative sign corresponds to a second asymptotically flat region, on the "other side" of the Einstein-Rosen bridge.

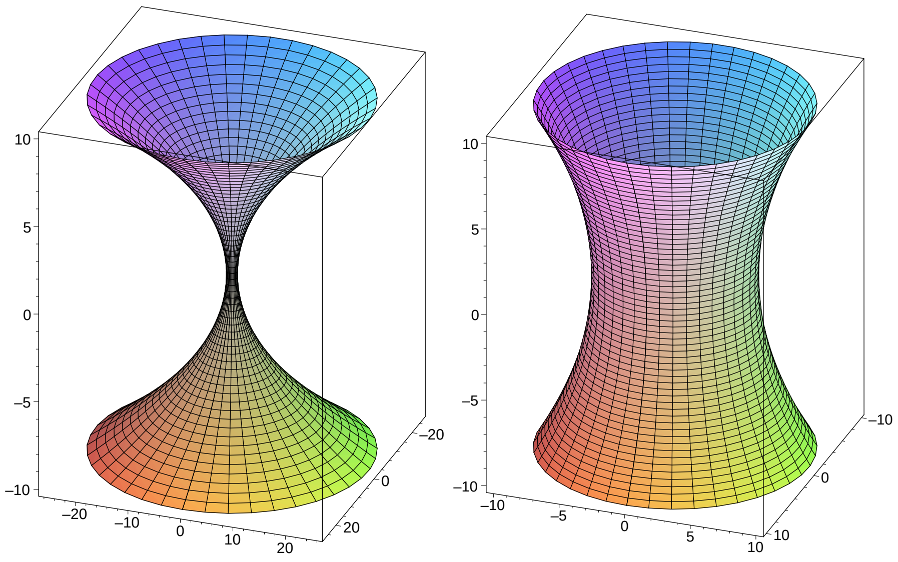

Figure 5.1: Isometric embedding of the space-geometry of an $n = 3$ dimensional Schwarzschild black hole into four-dimensional Euclidean space, near the throat of the Einstein-Rosen bridge $r = 2m$, with $2m = 1$ (left) and $2m = 6$ (right).

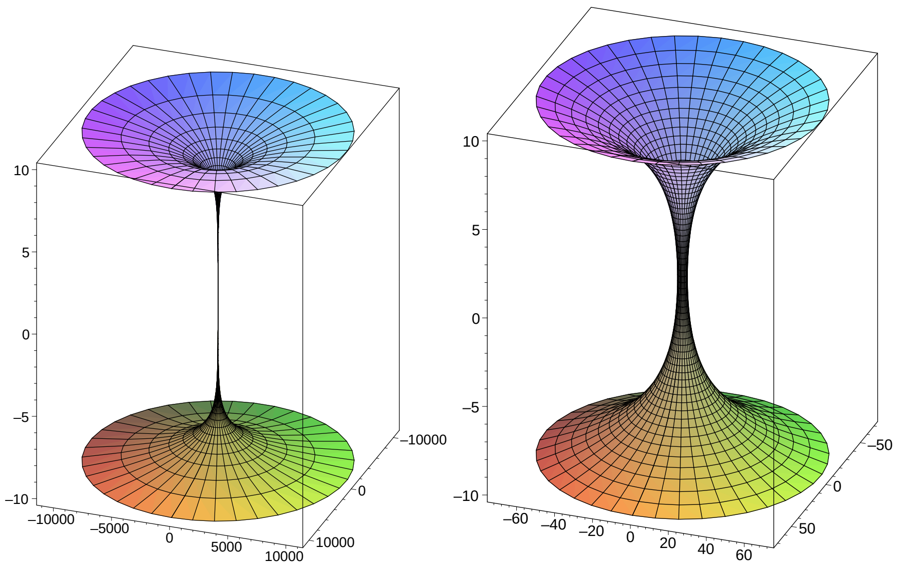

Figure 5.2: Isometric embedding of the space-geometry of an $n = 4$ dimensional Schwarzschild black hole into five-dimensional Euclidean space, near the throat of the Einstein-Rosen bridge $r = (2m)^{1/2}$, with $2m = 1$ (left) and $2m = 6$ (right). The extents of the vertical axis are the same as those in Figure 5.1.

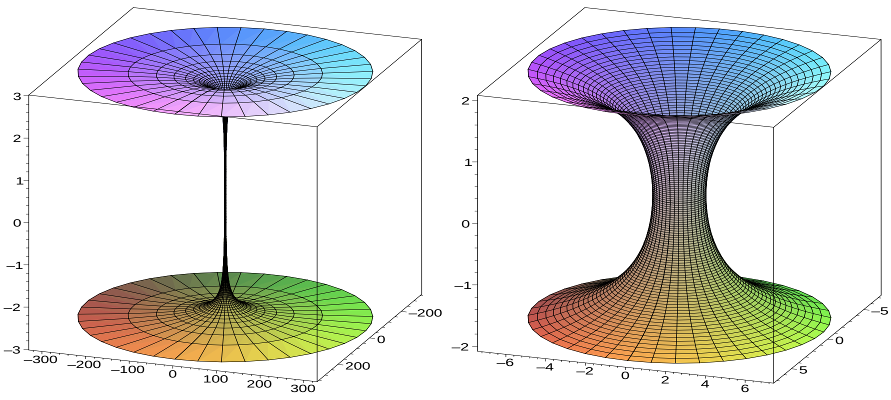

Figure 5.3: Isometric embedding of the *space-geometry* of a $(5+1)$–dimensional Schwarzschild black hole into six-dimensional Euclidean space, near the throat of the Einstein-Rosen bridge $r=(2m)^{1/3}$, with $2m=2$. The variable along the vertical axis asymptotes to $\approx \pm 3.06$ as r tends to infinity. The right picture is a zoom to the centre of the throat.

5.4 Geodesics

The Lagrangian for geodesics is:

$$\mathscr{L} = \frac{1}{2}\left(V^2\left(\frac{dt}{ds}\right)^2 - V^{-2}\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\theta}{ds}\right)^2 - r^2\sin^2\theta\left(\frac{d\varphi}{ds}\right)^2\right) .$$

The Euler-Lagrange equation for t reads

$$\frac{d}{ds}\left(V^2\frac{dt}{ds}\right) = 0 . \tag{5.4.1}$$

which gives a conservation law. Similarly

$$\frac{d}{ds}\left(r^2\sin^2\theta\frac{d\varphi}{ds}\right) = 0 . \tag{5.4.2}$$

The remaining equations are more complicated:

$$\frac{d}{ds}\left(V^{-2}\frac{dr}{ds}\right) = V\partial_r V\left(\frac{dt}{ds}\right)^2 + 2r\left[\left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{ds}\right)^2\right] \tag{5.4.3}$$

$$\frac{d}{ds}\left(r^2\frac{d\theta}{ds}\right) = r^2\sin\theta\cos\theta\left(\frac{d\varphi}{ds}\right)^2 . \tag{5.4.4}$$

As an exercice, one can read the Christoffel symbols of g from the above: e.g., from (5.4.1),

$$\Gamma^t{}_{tr} = \frac{V'}{V} ,$$

etc.

Since $\mathscr{L}$ is s-independent, the Hamiltonian H is conserved:

$$H := \frac{\partial \mathscr{L}}{\partial \dot{q}^\mu}\dot{q}^\mu - \mathscr{L} = \mathscr{L} \ .$$

This gives one more constant of motion

$$V^2\left(\frac{dt}{ds}\right)^2 - V^{-2}\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\theta}{ds}\right)^2 - r^2\sin^2\theta\left(\frac{d\varphi}{ds}\right)^2 = \lambda \ . \tag{5.4.5}$$

and without loss of generality, making an affine transformation of s, we can choose $\boxed{\lambda \in \{0, \pm 1\}}$.

To simplify things somewhat, let us start by showing that motion is *planar*. Consider any geodesic, and think of the coordinates (r, θ, φ) as spherical coordinates on $\mathbb{R}^3$. Then the initial position vector (which is assumed *not* to be the origin, as the metric is singular at $r = 0$) and the initial velocity vector, which is assumed *not* to be radial (otherwise the geodesic will be radial, and the claim is true) define a unique plane in $\mathbb{R}^3$. We can then choose the spherical coordinates so that this plane is the plane $\theta = \pi/2$. This leads to $\theta(0) = \pi/2$ and $\dot{\theta}(0) = 0$, and then the function $\theta(s) \equiv \pi/2$ is a solution of (5.4.8) satisfying the initial values. By uniqueness of solutions, this is *the* solution.

We have thus shown that $\boxed{\text{without loss of generality we can assume } \theta = \pi/2}$. In this case the equations of motion are

$$\frac{d}{ds}\left(V^2\frac{dt}{ds}\right) = 0 \quad \Longrightarrow \quad \frac{dt}{ds} = \frac{E}{1-\frac{2m}{r}} \ . \tag{5.4.6}$$

$$\frac{d}{ds}\left(r^2\frac{d\varphi}{ds}\right) = 0 \quad \Longrightarrow \quad \frac{d\varphi}{ds} = \frac{J}{r^2} \ . \tag{5.4.7}$$

$$\frac{d}{ds}\left(V^{-2}\frac{dr}{ds}\right) = V\partial_r V\left(\frac{dt}{ds}\right)^2 + 2r\left(\frac{d\theta}{ds}\right)^2 \ . \tag{5.4.8}$$

We will ignore this last equation, combining instead (5.4.6)-(5.4.7) with (5.4.5),

$$\underbrace{V^2\left(\frac{dt}{ds}\right)^2}_{E^2V^{-2}} - V^{-2}\left(\frac{dr}{ds}\right)^2 - \underbrace{r^2\left(\frac{d\varphi}{ds}\right)^2}_{J^2/r^2} = \lambda \in \{0, \pm 1\} \ . \tag{5.4.9}$$

to conclude that

$$\left(\frac{dr}{ds}\right)^2 = -\left(\lambda + \frac{J^2}{r^2}\right)(1 - \frac{2m}{r}) + E^2 \; . \tag{5.4.10}$$

The second order equation of motion could be obtained from (5.4.8), but at this stage it is simpler to realize that differentiation of (5.4.10) with respect to s *must* produce the second order equation of motion, obtaining

$$2\frac{d^2r}{ds^2} = \frac{d}{dr}\Big(E^2 - (1 - \frac{2m}{r})(\lambda + \frac{J^2}{r^2})\Big) \; . \tag{5.4.11}$$

For further reference we note the following: let $u = m/r$, from (5.4.10) and (5.4.7) one obtains, for $J \neq 0$,

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{m^2E^2}{J^2} - u^2(1 - 2u) - \frac{\lambda m^2(1 - 2u)}{J^2} \; , \tag{5.4.12}$$

where $\lambda = 0$ for null geodesics, and $\lambda = 1$ for timelike ones. (You might wish to compare this with a similar equation derived in Newtonian gravity.)

Similarly to (5.4.11), the second order equation of motion for u can be obtained by differentiating with respect to φ, leading to

$$2\frac{d^2u}{d\varphi^2} = \frac{d}{du}\left(-u^2(1 - 2u) - \frac{\lambda m^2(1 - 2u)}{J^2}\right) \; . \tag{5.4.13}$$

The interpretation of E

In this section we momentarily suspend the convention that $c = 1$.

Consider two observers in special relativity, with unit four-velocity vectors u^μ and v^μ, there exists a frame in which (u^μ) equals $(1, \vec{0}) = (1, 0, 0, 0)$, while $v^\mu\partial_\mu$ has the general form $(v^\mu) = (\gamma^{-1}, \gamma^{-1}\vec{v})$, so that the scalar product $u^\mu v_\mu$ equals

$$u^\mu v_\mu = \gamma^{-1} = \frac{1}{\sqrt{1 - |\vec{v}|^2 c^{-2}}} \; . \tag{5.4.14}$$

In curved space-time we can use local inertial frames of two observers passing through the same space-time point to similarly obtain (5.4.14), where $\vec{v}$ is the space-velocity of the local inertial frames with respect to each other.

We turn our attention now to a test observer moving in a Schwarzschild gravitational field. We want to relate the parameter m appearing there to Newtonian physics. To avoid any preconceived ideas, let us call this parameter α, writing thus the metric as

$$g = \left(1 - \frac{2\alpha}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{2\alpha}{r}} - r^2 d\Omega^2 \; .$$

Let u^μ be the four-velocity vector of the *stationary* observers of Section 5.2 (with m there replaced by α) hence, as seen in (5.2.2),

$$u^\mu \partial_\mu = \frac{1}{\sqrt{1-\frac{2\alpha}{r}}} \partial_t \; .$$

We then obtain, for a freely falling observer with $v^\mu \partial_\mu = \dot{x}^\mu \partial_\mu$,

$$\frac{1}{\sqrt{1-|\vec{v}|^2 c^{-2}}} = u^\mu v_\mu = g_{\mu\nu} u^\mu v^\mu = g_{00} \underbrace{u^0}_{1/\sqrt{1-\frac{2\alpha}{r}}} v^0 = \sqrt{1-\frac{2\alpha}{r}} \underbrace{\dot{t}}_{\frac{E}{1-\frac{2\alpha}{r}}} = \frac{E}{\sqrt{1-\frac{2\alpha}{r}}} \; .$$

Hence

$$E = \frac{\sqrt{1-\frac{2\alpha}{r}}}{\sqrt{1-|\vec{v}|^2 c^{-2}}} \; .$$

Taylor expanding, for large r and small v/c, and multiplying by the rest energy, say $m_0 c^2$, of the geodesic observer we obtain

$$E m_0 c^2 \approx m_0 c^2 + \frac{m_0}{2}|\vec{v}|^2 - \frac{m_0 \alpha c^2}{r} \; . \tag{5.4.15}$$

This leads to the obvious interpretation of each term, as rest energy, Newtonian kinetic energy, and Newtonian gravitational energy in a gravitational field generated by a spherically symmetric object of mass

$$m = \frac{\alpha c^2}{G} \; .$$

In other words, for m to be the Newtonian mass seen at large distances, the Schwarzschild metric should read

$$g = \left(1 - \frac{2Gm}{c^2 r}\right) dt^2 - \frac{dr^2}{1-\frac{2Gm}{c^2 r}} - r^2 d\Omega^2 \; .$$

Hence, in units where $G = c = 1$, we conclude that an interpretation of E as the *general relativistic energy per unit mass of the geodesic test observers* is consistent with the Newtonian limit. The property that E is constant along geodesics is then the law of conservation of energy (per unit mass) for freely falling observers in a Schwarzschild field.

5.4.1 Photons

In special relativity photons move along straight lines with null tangent $\eta(\dot\gamma, \dot\gamma) = 0$; these are affinely parameterized geodesics of the Minkowski metric η. In view of the correspondence principle we require that

$$\boxed{\textit{test photons in general relativity move along null geodesics.}}$$

Here, a *test photon* is a photon, the gravitational field of which can be ignored at the scale at which experiments are carried out.

Periodic null geodesics

We will not attempt an exhaustive analysis of photon trajectories, but only point out a few basic facts. We start by noting the existence of a striking class of *null* geodesics for which $r(s) = \text{const}$. It follows from (5.4.11), and from uniqueness of solutions of the Cauchy problem for ODE's, that such a curve will be a null geodesic provided that the right-hand-sides of (5.4.10) and of (5.4.11) (with $\lambda = 0$) vanish:

$$E^2 - (1 - \frac{2m}{r})\frac{J^2}{r^2} = 0 = \frac{2J^2}{r^3}(-r + 3m) \ . \tag{5.4.16}$$

Simple algebra shows then that the curves

$$s \mapsto \gamma_\pm(s) = (t = s,\ r = 3m,\ \theta = \pi/2,\ \varphi = \pm 3^{3/2} m^{-1} s) \ ,$$

are null geodesics spiraling on the timelike cylinder $\{r = 3m\}$.

Gravitational redshift

In this section we wish to derive the frequency shift along radial null geodesics. So let a wave of light with frequency ω_1 be emitted radially at r_1, and let

$$\Delta s_1 := 1/\omega_1$$

be the proper time between two consecutive maxima of the wave. In the "geometric optics approximation", the wave then travels outwards on radial null geodesics.

On such geodesics we have $\lambda = 0$, while θ and ϕ are constant. Therefore

$$\left(1 - \frac{2m}{r}\right) \ dt^2 - \frac{dr^2}{1 - 2m/r} = 0 \,,$$

leading to

$$\frac{dt}{dr} = \frac{r}{r - 2m} \ . \tag{5.4.17}$$

By integrating (5.4.17) we find that the coordinate time t_1 at which the crest of the wave leaves C_1 is related to the coordinate time t_2 at which it arrives at C_2 by

$$t_2 - t_1 = \int_{r_1}^{r_2} \frac{r \ dr}{r - 2m} \ . \tag{5.4.18}$$

Since the right-hand side is of (5.4.18) is independent of t_1, we have that the coordinate time interval Δt_1 between the emission times of two successive crests at r_1 is the same as the coordinate time interval Δt_2 between their observations by a stationary observer at r_2.

Recall that the four-velocity vector

$$u \equiv u^\mu \partial_\mu := \frac{dx^\mu}{ds} \partial_\mu$$

of a stationary observer at coordinate radius r takes the form (5.2.2):

$$u = \frac{1}{\sqrt{1-\frac{2m}{r}}}\partial_t = \frac{dt}{ds}\partial_t \quad \Longleftrightarrow \quad \frac{dt}{ds} = \frac{1}{\sqrt{1-\frac{2m}{r}}} . \tag{5.4.19}$$

Since the emitter of light has been assumed to be stationary, (5.4.19) shows that the proper time interval Δs_1 is related to the coordinate time interval Δt_1 by

$$\Delta t_1 = \frac{dt}{ds}\Delta s_1 = \frac{1}{\sqrt{1-\frac{2m}{r_1}}}\Delta s_1 ,$$

with a similar formula relating Δt_2 with Δs_2.

As we have seen that $\Delta t_1 = \Delta t_2$, we obtain

$$\frac{\Delta s_1}{\Delta s_2} = \frac{\sqrt{1-2m/r_1}}{\sqrt{1-2m/r_2}} . \tag{5.4.20}$$

Subsequently,

$$\omega_2 = \frac{\sqrt{1-\frac{2m}{r_1}}}{\sqrt{1-\frac{2m}{r_2}}}\omega_1 . \tag{5.4.21}$$

This is the *gravitational redshift formula*, as observed e.g. in the Pound-Rebka experiment, see e.g. `http://en.wikipedia.org/wiki/Pound-Rebka_experiment`. We see that the frequency observed by an observer at infinity will be smaller than the energy emitted at any finite radius. More generally, if $r_2 > r_1$, then the observed spectrum will be shifted to the red, by a frequency-independent multiplicative factor, as compared to the emitted one.

As another application of (5.4.21), imagine that you send a beacon emitting at constant frequency towards a black-hole. You will see its frequency shifting away towards the red as the beacon approaches the event horizon $r = 2m$, with the frequency tending to zero as the event horizon is approached.

An alternative derivation of the redshift effect uses the *frequency four vector* K of a photon, seen in the special relativity lectures: If a photon moves along a Minkowski space-time null geodesic γ, then K is a constant multiple of $\dot\gamma$ such that an observer moving with four-velocity U observes a frequency ω given by

$$\omega = U^\mu K_\mu .$$

In general relativity the frequency vector will thus again be a constant multiple of the tangent $\dot\gamma$ to a null geodesic.

So, consider a static observer O_1 at $r = r_1$, with velocity four-vector U_1, which sends a photon with frequency ω_1 to a static observer O_2 at $r = r_2$, with velocity four-vector U_2. We want to find the frequency ω_2 observed by O_2. If $K = \lambda\dot\gamma$, for some constant λ,[•5.4.1] then the frequencies at O_a, $a = 1, 2$, equal

•5.4.1: *not to be confused with the wave length! poor notation here...*

$$\omega_a = g(U_a, K) = g_{00}U_a^0 K^0 = \lambda g_{00}U_a^0 \dot t = \lambda\sqrt{1-\frac{2m}{r_a}}\dot t\ .$$

But $(1 - 2m/r)\dot t$ is constant along the null geodesic followed by the photon. So

$$\omega_2 = \lambda\sqrt{1-\frac{2m}{r_2}}\dot t = \lambda\frac{1}{\sqrt{1-\frac{2m}{r_2}}}\underbrace{\left(1-\frac{2m}{r_2}\right)\dot t}_{=\left(1-\frac{2m}{r_1}\right)\dot t} = \frac{\sqrt{1-\frac{2m}{r_1}}}{\sqrt{1-\frac{2m}{r_2}}}\underbrace{\lambda\sqrt{1-\frac{2m}{r_1}}\dot t}_{\omega_1}\ ,$$

and we have recovered (5.4.21).

Weak field light bending

For null geodesics (5.4.13) reads

$$\frac{d^2u}{d\varphi^2} = -u + 3u^2\ . \qquad (5.4.22)$$

For u very small or, equivalently, for r large as compared to m, an excellent approximation is obtained by neglecting the quadratic term, leading to

$$u_0 = \alpha\cos(\varphi - \varphi_0)\ ,$$

for some (small) constant $\alpha \neq 0$ (otherwise $r = \infty$). By a redefinition of φ we can always achieve $\varphi_0 = 0$. Equivalently,

$$\alpha r\cos\varphi = m\ ,$$

which is the equation for the straight line $x = m/\alpha$ in the (x, y) plane.

We can calculate the leading order correction to this by writing $u = \alpha\cos\varphi + v(\varphi)$, where $v = O(\alpha^2)$ is small. Inserting into (5.4.22) and neglecting terms which are $O(\alpha^3)$ one obtains

$$v'' + v = 3\alpha^2\cos^2\varphi\ .$$

This is easily integrated to give

$$v = A\cos\varphi + B\sin\varphi + \alpha^2(1 + \sin^2\varphi)\ .$$

We choose A and B so that at $\varphi = 0$ the initial data for the orbit coincide with those for the unperturbed one,

$$u(0) = \alpha = \frac{m}{d}\ , \quad \dot u(0) = 0\ ,$$

where d is the distance of the closest approach of the orbit to the origin $r = 0$. This gives $A = -\alpha^2$, $B = 0$, and

$$u = \underbrace{(\alpha - \alpha^2)\cos\varphi + \alpha^2(1 + \sin^2\varphi)}_{=:u_1} + O(\alpha^3) \ .$$

Now, the "Newtonian" orbit u_0 was a straight line. By definition of u, we have $r \to \infty$ if and only if $u \to 0$; for u_0 this corresponds to $\varphi \to \pm\pi/2$. However, the corrected orbit u_1 will reach zero at angles $\pm\varphi_\alpha = \pm(\pi/2 + \gamma_\alpha)$, slightly larger in modulus than $\pi/2$:

$$0 = \frac{u_1(\varphi_\alpha)}{\alpha} = (1-\alpha)\cos\varphi_\alpha + \alpha(1+\sin^2\varphi_\alpha) = -(1-\alpha)\sin\gamma_\alpha + \alpha(1+\cos^2\gamma_\alpha) \ .$$

Approximating $\sin\gamma_\alpha$ by γ_α, and $\cos\gamma_\alpha$ by one, one obtains

$$\gamma_\alpha = 2\alpha + O(\alpha^2) \ .$$

The total bending of the orbit is 4α, giving the final SI formula for the angle deflection

$$\frac{4mG}{dc^2} \ ,$$

recall that d is the distance of closest approach to the center.

For a light ray just grazing the surface of the sun, so that $m = M_☼$, $d = r_♁$, one obtains a deflection of 10^{-5} radians or $2''$. This effect was claimed to have been observed by Eddington during the 1919 eclipse expedition to Africa, by comparing photographs of the star field near the sun during an eclipse with a photograph of the same star field when the sun was not interfering. It should, however, be said that some researchers have expressed doubts about the reliability of the conclusions that could have been drawn from the data available at the time.

5.4.2 Massive test particles

Perihelion/periastron precession

Given a Keplerian orbit around a star, the *periastron* is the point at which the orbit is closest to the star. When the star is our sun, this point is usually called the *perihelion*.

We want to compare the motion in the Schwarzschild metric with the Keplerian orbits, at distances large compared to m. Here the following should be kept in mind: for the orbit of the earth (so that $r = r_♁$ and $J = J_♁$) around the sun (so that $m = M_☼$) we have

$$\frac{2M_☼}{r_♁} = 2u \sim 10^{-8} \ , \quad \frac{J_♁^2}{r_♁^2} = J_♁^2 M_☼^{-2} u^2 \sim 10^{-8} \ , \quad \frac{M_☼}{J_♁^2} \sim 10^{-8} \ .$$

(Recall that m there is the mass of the central body, while J is the angular momentum per unit mass of the orbiting one.)

So we return to (5.4.10) with $\lambda = 1$

$$\left(\frac{dr}{ds}\right)^2 = -\left(1+\frac{J^2}{r^2}\right)(1-\frac{2m}{r}) + E^2 = E^2 - 1 - \frac{J^2}{r^2} + \frac{2m}{r} + \frac{2mJ^2}{r^3}\ . \quad (5.4.23)$$

and we consider solutions which are perturbations of the the corresponding Newtonian solutions. Now, for a body with unit mass, and with angular momentum J, conservation of angular-momentum leads to the following formula for the Newtonian energy E_N

$$E_N = \frac{1}{2}(\dot r^2 + r^2\dot\varphi^2) - \frac{m}{r} = \frac{1}{2}(\dot r^2 + \frac{J^2}{r^2}) - \frac{m}{r}\ ,$$

hence

$$\left(\frac{dr}{ds}\right)^2 = 2E_N - \frac{J^2}{r^2} + \frac{2m}{r}\ . \quad (5.4.24)$$

So we obtain the same equations if we neglect the $\frac{2mJ^2}{r^3}$ term in (5.4.23), and identify E^2-1 with $2E_N$; as already discussed, this last identification is consistent with (5.4.15) at points where m/r and $|\vec v|^2$ are small compared to one (in units where $G = c = 1$).

Replacing r by $u = m/r$, we conclude that (5.4.12) with $\lambda = 1$,

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{m^2E^2}{J^2} - u^2(1-2u) - \frac{m^2(1-2u)}{J^2}\ , \quad (5.4.25)$$

should be compared with its version where the u^3 terms have been dropped out:

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{m^2E^2}{J^2} - u^2 - \frac{m^2(1-2u)}{J^2} = \frac{m^2}{J^2}(E^2-1+2u) - u^2$$

$$\Longleftrightarrow \left(\frac{d(u-\frac{m^2}{J^2})}{d\varphi}\right)^2 + (u-\frac{m^2}{J^2})^2 = \frac{m^4}{J^4}\underbrace{(1+\frac{J^2(E^2-1)}{m^2})}_{=:e^2}\ . \quad (5.4.26)$$

Consider, then the Newtonian solution

$$u_0 = \frac{m^2}{J^2}(1+e\cos\varphi)\ , \qquad e^2 = 1 + \frac{J^2(E^2-1)}{m^2}\ , \quad (5.4.27)$$

which has been chosen so that $\dot u = 0$ at $\varphi = 0$. The equation of motion (5.4.13) for u reads

$$\frac{d^2u}{d\varphi^2} + u = 3u^2 + \frac{m^2}{J^2}\ . \quad (5.4.28)$$

We write $u = u_0 + v$, where v is small, and insert in (5.4.28) to obtain

$$\frac{d^2 v}{d\varphi^2} + v = 3\frac{m^4}{J^4}(1 + e\cos\varphi)^2 \; . \tag{5.4.29}$$

This can be integrated to obtain

$$v(\varphi) = \frac{m^4}{J^4}\left[-(3+e^2)\cos\varphi + 3\left(1+\frac{e^2}{2}\right) - \frac{e^2}{2}\cos 2\varphi + 3e\varphi\sin\varphi\right] \; ,$$

where the free integration constants have been chosen so that $v(0) = v'(0) = 0$. Thus,

$$u(\varphi) \approx \frac{m^2}{J^2}(1+e\cos\varphi) + \frac{m^4}{J^4}\left[-(3+e^2)\cos\varphi + 3\left(1+\frac{e^2}{2}\right) - \frac{e^2}{2}\cos 2\varphi + 3e\varphi\sin\varphi\right] \; .$$

As already mentioned, the *perihelion* is the point of closest approach to the center, hence a maximum of u. At $\varphi = 0$ we have $\partial_\varphi u = 0$, so this is indeed an extremum, and it is clear from the Newtonian expression (5.4.27) that this is a maximum of u for m/J small enough. The next maximum will be at $\varphi = 2\pi + \gamma$, with γ small:

$$\begin{aligned} 0 &= \partial_\varphi u \approx -\frac{m^2}{J^2}e\sin\varphi + \frac{m^4}{J^4}\Big[\underbrace{(3+e^2)\sin\varphi + e^2\sin 2\varphi + 3e\sin\varphi}_{\approx 0} + 3e\varphi\cos\varphi\Big] \\ &\approx -\frac{m^2}{J^2}e\gamma + 6\frac{m^4 e\pi}{J^4} \; . \end{aligned}$$

We have thus obtained, in SI units,

$$\gamma \approx 6\frac{m^2\pi G}{J^2 c^2} \; .$$

This is the *perihelion advance* predicted by general relativity. For Mercury this is sometimes called the *perihermion* advance, and equals about

$$40'' \text{ per century} \; .$$

A more detailed calculation gives the observed value of $43''$ per century. This value was known to astronomers at the beginning of the twentieth century, one suggested explanation being the existence of another planet between Mercury and the sun.

For the Taylor-Hulse pulsar, described in the first lecture, the advance is around 4^o per year. This has to be corrected to account for the gravitational waves emitted by the system; the observed corrections agree extremely well with the theory, and provide an indirect proof of existence of gravitational waves.

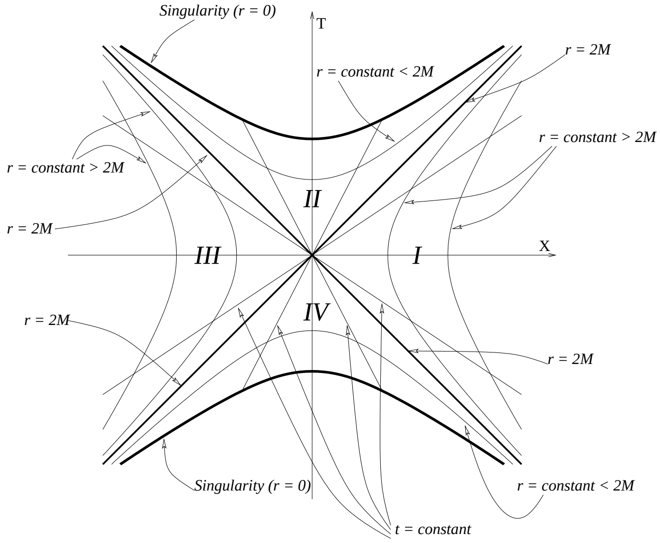

Figure 5.4: The Kruskal-Szekeres extension of the Schwarzschild solution.

5.5 The Kruskal-Szekeres extension

The transition from

$$g = (1 - \tfrac{2m}{r})dt^2 - \tfrac{dr^2}{1-\frac{2m}{r}} - r^2 d\Omega^2 \tag{5.5.1}$$

to

$$g = (1 - \frac{2m}{r})dv^2 - 2dvdr - r^2 d\Omega^2 \tag{5.5.2}$$

using the coordinate

$$v = t + f(r) \ , \quad f' = \frac{1}{1 - \frac{2m}{r}} \ , \tag{5.5.3}$$

so that

$$v = t + r + 2m \ln(r - 2m) \ ,$$

is not the end of the story, as further extensions are possible, which will be clear from the calculations that we will do shortly. For the metric (5.5.1) a maximal analytic extension has been found independently by Kruskal [3], Szekeres [7], and Fronsdal [2]; for some obscure reason Fronsdal is almost never mentioned in this context. This extension is visualised[1] in Figure 5.4. The region I there corresponds to $r > 2m$, while the extension constructed using the (v, r, θ, φ) coordinates corresponds to the regions I and II.

The general construction, for spherically symmetric metrics, which we write in the form

$$g = V^2 dt^2 - \tfrac{dr^2}{V^2} - r^2 d\Omega^2 \ , \tag{5.5.4}$$

[1] I am grateful to J.-P. Nicolas for allowing me to use his electronic figures [5].

proceeds as follows: We introduce another coordinate u defined by changing a sign in the previous equation for v,

$$v = t + f(r) \ , \quad f' = \frac{1}{V^2} \ , \tag{5.5.5}$$

to

$$u = t - f(r) \ , \quad f' = \frac{1}{V^2} \ , \tag{5.5.6}$$

so that

$$u = t - r - 2m \ln(r - 2m)$$

in the Schwarzschild case.

Similarly to what we have done in a previous lecture, one can now replace (t, r) by (u, r), obtaining an extension of the exterior region I of Figure 5.4 into the "white hole" region IV. However, it is easier to understand this picture by passing directly to the complete extension, which proceeds in two steps. First, we replace (t, r) by (u, v). We note that

$$V\, du = V\, dt - \frac{1}{V} dr \ , \quad V\, dv = V\, dt + \frac{1}{V} dr \ ,$$

which gives

$$V\, dt = \frac{V}{2}(du + dv) \ , \quad \frac{1}{V} dr = \frac{V}{2}(dv - du) \ .$$

Inserting this into (5.5.4) brings g to the form

$$\begin{aligned} g &= V^2 dt^2 - V^{-2} dr^2 - r^2 d\Omega^2 \\ &= \frac{V^2}{4}\Big((du + dv)^2 - (du - dv)^2\Big) - r^2 d\Omega^2 \\ &= V^2 du\, dv - r^2 d\Omega^2 \ . \end{aligned} \tag{5.5.7}$$

The metric so obtained is still degenerate at $\{V = 0\}$. The desingularisation is now obtained by setting

$$\hat{u} = -\exp(-cu) \ , \quad \hat{v} = \exp(cv) \ , \tag{5.5.8}$$

with an appropriately chosen c: since

$$d\hat{u} = c \exp(-cu)\, du \ , \quad d\hat{v} = c \exp(cv) dv \ ,$$

we obtain

$$\begin{aligned} V^2 du\, dv &= \frac{V^2}{c^2} \exp(c(u - v)) d\hat{u}\, d\hat{v} \\ &= \frac{V^2}{c^2} \exp(-2cf(r)) d\hat{u}\, d\hat{v} \ . \end{aligned}$$

In the Schwarzschild case this reads

$$\begin{aligned}\frac{V^2}{c^2}\exp(2cf(r)) &= \frac{r-2m}{c^2 r}\exp(-2c(r+2m\ln(r-2m))) \\ &= \frac{\exp(-2cr)}{c^2 r}(r-2m)\exp(-4mc\ln(r-2m))\ ,\end{aligned}$$

and with the choice

$$4mc = 1$$

the term $r-2m$ cancels out, leading to a factor in front of $d\hat{u}\,d\hat{v}$ which has no zeros near $r = 2m$. Thus, the desired coordinate transformation is

$$\hat{u} = -\exp(-cu) = -\exp(\tfrac{r-t}{4m})\sqrt{r-2m}\ , \tag{5.5.9}$$

$$\hat{v} = \exp(cv) = \exp(\tfrac{r+t}{4m})\sqrt{r-2m}\ , \tag{5.5.10}$$

with g taking the form

$$\begin{aligned} g &= V^2 du\,dv - r^2 d\Omega^2 \\ &= \frac{16m^2\exp(-\frac{r}{2m})}{r} d\hat{u}\,d\hat{v} - r^2 d\Omega^2\ . \end{aligned} \tag{5.5.11}$$

Here r should be viewed as a function of $\hat{u}$ and $\hat{v}$ defined implicitly by the equation

$$-\hat{u}\hat{v} = \underbrace{\exp(\frac{r}{2m})(r-2m)}_{=:G(r)}\ . \tag{5.5.12}$$

Indeed, we have

$$\Big(\exp(\frac{r}{2m})(r-2m)\Big)' = \frac{r}{2m}\exp(\frac{r}{2m}) > 0\ ,$$

which shows that the function G defined at the right-hand-side of (5.5.12) is a smooth strictly increasing function of $r>0$. We have $G(0) = -2m$, and G tends to infinity as r does, so G defines a bijection of $(0,\infty)$ with $(-2m,\infty)$. The implicit function theorem guarantees smoothness of the inverse G^{-1}, and hence the existence of a smooth function $r = G^{-1}(-\hat{u}\hat{v})$ solving (5.5.12) on the set $\hat{u}\hat{v} \in (-\infty, 2m)$.

Note that so far we had $r > 2m$, but there are *a priori* no reasons for the function $r(u,v)$ defined above to satisfy this constraint. In fact, we already know from our experience with the (v, r, θ, φ) coordinate system that a restriction $r > 2m$ would lead to a space-time with poor global properties.

We have $\det g = -\frac{\exp(-\frac{r}{m})}{(16)^2 m^4} r^2 \sin^2\theta$, with all coefficients of g smooth, which shows that (5.5.11) defines a smooth Lorentzian metric on the set

$$\hat{u},\ \hat{v} \in \mathbb{R}\ , \quad r > 0\ . \tag{5.5.13}$$

This is the *Kruszkal-Szekeres* extension of the original Schwarzschild space-time. Figure 5.4 gives a representation of the extended space-time in coordinates

$$X = (\hat{v} - \hat{u})/2 , \quad T = (\hat{v} + \hat{u})/2 .$$

Here one should keep in mind that, as already showns, the metric cannot be extended across the set $r = 0$ in the class of C^2 metrics.

Let us discuss some features of Figure 5.4:

1. The singular set $r = 0$ corresponds to the spacelike hyperboloids

$$(X^2 - T^2)|_{r=0} = -\hat{u}\hat{v}|_{r=0} = 2m > 0 .$$

2. More generally, the sets $r = \text{const}$ are hyperboloids $X^2 - T^2 = \text{const}'$, which are timelike in the regions I and III (since $X^2 - T^2 < 0$ there), and which are spacelike in the regions II and IV.

3. The vector field ∇T satisfies

$$g(\nabla T, \nabla T) = g^\sharp(dT, dT) = \frac{1}{4} g^\sharp(d\hat{u} + d\hat{v}, d\hat{u} + d\hat{v}) = \frac{1}{2} g^\sharp(d\hat{u}, d\hat{v}) < 0 ,$$

   which shows that T is a time coordinate. Similarly X is a space-coordinate, so that Figure 5.4 respects our implicit convention of representing time along the vertical axis and space along the horizontal one.

4. The map

$$(\hat{u}, \hat{v}) \to (-\hat{u}, -\hat{v})$$

   is clearly an isometry, so that the region I is isometric to region III, and region II is isometric to region IV. In particular the extended manifold has two asymptotically flat regions, the original region I, and region III which is an identical copy I.

5. The hypersurface $t = 0$ from the region I corresponds to $\hat{u} = -\hat{v} > 0$, equivalently it is the subset $X > 0$ of the hypersurface $T = 0$. This can be smoothly continued to negative X, which corresponds to a second copy of this hypersurface. The resulting geometry is often referred to as the *Einstein-Rosen bridge*. It is instructive to do the continuation directly using the Riemannian metric γ induced by g on $t = 0$:

$$\gamma = \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega^2 , \quad r > 2m .$$

   A convenient coordinate ρ is given by

$$\rho = \sqrt{r^2 - 4m^2} \iff r = \sqrt{\rho^2 + 4m^2} .$$

This brings γ to the form

$$\gamma = \Big(1 + \frac{2m}{\sqrt{\rho^2 + 4m^2}}\Big) d\rho^2 + (\rho^2 + 4m^2) d\Omega^2 \ , \qquad (5.5.14)$$

which can be smoothly continued from the original range $\rho > 0$ to $\rho \in \mathbb{R}$. Equation (5.5.14) further exhibits explicitly asymptotic flatness of both asymptotic regions $\rho \to \infty$ and $\rho \to -\infty$. Indeed,

$$g \sim d\rho^2 + \rho^2 d\Omega^2$$

to leading order, for large $|\rho|$, which is the flat metric in radial coordinates with radius $|\rho|$.

6. In the Kruskal-Szekeres coordinate system the Killing vector field $K = \partial_t$ takes the form

$$\begin{aligned} K &= \partial_t = \frac{\partial \hat u}{\partial t}\partial_{\hat u} + \frac{\partial \hat v}{\partial t}\partial_{\hat v} \\ &= -\hat u \partial_{\hat u} + \hat v \partial_{\hat v} \ . \end{aligned} \qquad (5.5.15)$$

More precisely, the Killing vector field ∂_t defined on the original Schwarzschild region extends to a Killing vector field K defined throughout the Kruskal-Szekeres manifold by the right-hand-side of (5.5.15).

We note that K is tangent to the level sets of $\hat u$ or $\hat v$ at $\hat u \hat v = 0$, and therefore is null there. Moreover, it vanishes at the sphere $\hat u = \hat v = 0$, which is called *the bifurcation surface of the horizon*. The justification of this last terminology should be clear from Figure 5.4. Quite generally, a hypersurface to which a Killing vector is tangent, and null there, is called a *Killing horizon*. Therefore the union $\{\hat u \hat v = 0\}$ of the black hole horizon $\{\hat u = 0\}$ and the white hole event horizon $\{\hat v = 0\}$ can be written as the union of four Killing horizons and of their bifurcation surface.

The bifurcate horizon structure, as well as the formula (5.5.15), are rather reminiscent of what happens when considering the Killing vector $t\partial_x + x\partial_t$ in Minkowski space-time; this is left as an exercice to the reader.

The Kruskal-Szekeres extension is *inextendible*, which can be proved as follows: first, (5.1.5) shows that the *Kretschmann scalar* $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ diverges as r approaches zero. As already pointed out, this implies that no C^2 extension of the metric is possible across the set $\{r = 0\}$. Next, an analysis of the geodesics of the Schwarzschild metric shows that all (maximally extended) geodesics which *do not* approach $\{r = 0\}$ are complete. This implies inextendibility.

It can be shown that the Kruskal-Szekeres extension is singled out by being maximal in the vacuum, analytic, simply connected class, with all maximally extended geodesics γ either complete, or with the curvature scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ diverging along γ in finite affine time.

5.6 Conformal Carter-Penrose diagrams

Consider a metric with the following product structure:

$$g = \underbrace{g_{rr}(t,r)dr^2 + 2g_{rt}(t,r)dtdr + g_{tt}(t,r)dr^2}_{=:{}^2g} + \underbrace{h_{AB}(t,r,x^A)dx^Adx^B}_{=:h} , \tag{5.6.1}$$

where h is Riemannian metric in dimension $n-1$. Then any causal vector for g is also a causal vector for ${}^2g$, and drawing light-cones for ${}^2g$ gives a good idea of the causal structure of $(\mathscr{M}, g)$. We have already done that in Figure 5.4 to depict the black hole character of the Kruskal-Szekeres space-time.

Now, it is not too difficult to prove that any two-dimensional Lorentzian metric can be brought locally to the form

$${}^2g = 2g_{uv}(u,v)dudv = 2g_{uv}(-dt^2 + dr^2) \tag{5.6.2}$$

in which the light-cones have slopes one, just as in Minkowski space-time. When using such coordinates, it is sufficient to draw their domain of definition to visualise the global causal structure of the space-time.

Exercice 5.6.1 Prove (5.6.2). (Hint: use coordinates associated with right-going and left-going null geodesics.)

The above are the first two-ingredients behind the idea of conformal Carter-Penrose diagrams. The last thing to do is to bring any infinite domain of definition of the (u,v) coordinates to a finite one. For this, let $\bar u$ and $\bar v$ be defined by the equations

$$\tan \bar u = \frac{\hat u}{\sqrt{2m}} , \quad \tan \bar v = \frac{\hat v}{\sqrt{2m}} ,$$

where $\hat v$ and $\hat u$ have been defined in (5.5.9)-(5.5.10). Using

$$d\hat u = \frac{\sqrt{2m}}{\cos^2 \bar u} d\bar u , \quad d\hat v = \frac{\sqrt{2m}}{\cos^2 \bar v} d\bar v ,$$

the Schwarzschild metric takes the form

$$\begin{aligned} g &= \frac{16m^2 \exp(-\frac{r}{2m})}{r} d\hat u \, d\hat v - r^2 d\Omega^2 \\ &= \frac{32m^3 \exp(-\frac{r}{2m})}{r \cos^2 \bar u \cos^2 \bar v} d\bar u \, d\bar v - r^2 d\Omega^2 . \end{aligned} \tag{5.6.3}$$

Introducing new time- and space-coordinates $\bar t = (\bar u + \bar v)/2$, $\bar x = (\bar u - \bar v)/2$, so that

$$\bar u = \bar t - \bar x , \quad \bar v = \bar t + \bar x ,$$

one obtains a more familiar-looking form

$$g = \frac{32m^3 \exp(-\frac{r}{2m})}{r \cos^2 \bar{u} \cos^2 \bar{v}}(d\bar{t}^2 - d\bar{x}^2) - r^2 d\Omega^2 .$$

This is regular except at $\cos\bar{u} = 0$, and $\cos\bar{v} = 0$, and $r = 0$. The first set corresponds to the straight lines $\bar{u} = \bar{t} - \bar{x} \in \{\pm\pi/2\}$, while the second is the union of the lines $\bar{v} = \bar{t} + \bar{x} \in \{\pm\pi/2\}$.

The analysis of $\{r = 0\}$ requires some work: recall that $r = 0$ corresponds to $\hat{u}\hat{v} = 2m$, which is equivalent to

$$\tan(\bar{u})\tan(\bar{v}) = 1 .$$

Using the formula

$$\tan(\bar{u} + \bar{v}) = \frac{\tan\bar{u} + \tan\bar{v}}{1 - \tan\bar{u}\tan\bar{v}}$$

we obtain "$\tan(\bar{u} + \bar{v}) = \pm\infty$" or, more precisely,

$$\bar{u} + \bar{v} = 2\bar{t} = \pm\pi/2 .$$

So the Kruskal-Szekeres metric is conformal to a smooth Lorentzian metric on $C \times S^2$, where C is the set of Figure 5.5.

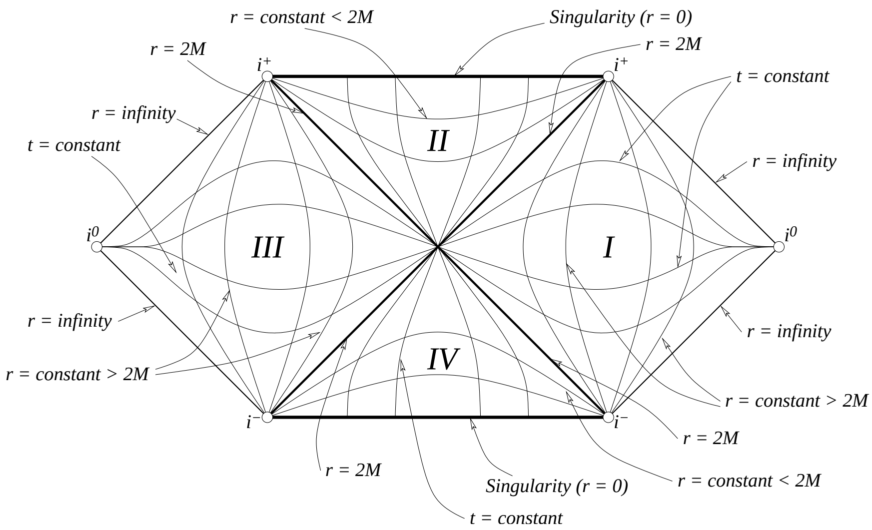

Figure 5.5: The Carter-Penrose diagram[1] for the Kruskal-Szekeres space-time with mass M. There are actually two asymptotically flat regions, with corresponding event horizons defined with respect to the second region. Each point in this diagram represents a two-dimensional sphere, and coordinates are chosen so that light-cones have slopes plus minus one. Regions are numbered as in Figure 5.4.

Bibliography

[1] A.L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 10, Springer Verlag, Berlin, New York, Heidelberg, 1987.

[2] C. Fronsdal, *Completion and embedding of the Schwarzschild solution*, Phys. Rev. (2) **116** (1959), 778–781. MR MR0110524 (22 #1402)

[3] M.D. Kruskal, *Maximal extension of Schwarzschild metric*, Phys. Rev. (2) **119** (1960), 1743–1745. MR MR0115757 (22 #6555)

[4] C.W. Misner, K. Thorne, and J.A. Wheeler, *Gravitation*, Freeman, San Fransisco, 1973.

[5] J.-P. Nicolas, *Dirac fields on asymptotically flat space-times*, Dissertationes Math. (Rozprawy Mat.) **408** (2002), 1–85. MR 1 952 742

[6] A.G. Riess *et al.*, *New Hubble Space Telescope discoveries of type Ia Supernovae at $z > 1$: Narrowing constraints on the early behavior of dark energy*, Astroph. Jour. **659** (2007), 98–121, arXiv:astro-ph/0611572.

[7] Gy. Szekeres, *On the singularities of a Riemannian manifold*, Gen. Rel. Grav. **34** (2002), 2001–2016, Reprinted from Publ. Math. Debrecen **7** (1960), 285–301 [MR0125541 (23 #A2842)]. MR MR1945497 (2003j:83073a)

[8] W. Michael Wood-Vasey et al., *Observational Constraints on the Nature of the Dark Energy: First Cosmological Results from the ESSENCE Supernova Survey*, Astroph. Jour. **666** (2007), 694–715, arXiv:astro-ph/0701041.

[9] N. M. J. Woodhouse, *General relativity*, Springer Undergraduate Mathematics Series, Springer-Verlag London Ltd., London, 2007. MR MR2268691 (2007i:83003)