

1 Some geodesics

Using the variational principle for geodesics, write down the geodesic equations, and give the obvious constants of motion (compare PS11 Q1ii and Q2ii), for

a) the *Schwarzschild* metric

$$g_m = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \quad (1)$$

(you might wish to do the calculation for a metric of the form

$$-e^{2f(r)} dt^2 + e^{-2f(r)} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2),$$

and specialize the result to Schwarzschild at the end, as many other important metrics are of this form, so the result might be useful later). Show that a geodesic initially tangent to the equatorial plane always remains in it.

b) The following *pp-wave* metric

$$g = dx^2 + dy^2 - 2du dv + H(u, x) du^2 .$$

c) The “post-Newtonian” metric

$$g_{00} = -\left(1 - \frac{2GM}{r}\right), \quad g_{0i} = 0, \quad g_{ij} = \left(1 + \frac{2GM}{r}\right) \delta_{ij},$$

with  $i, j \in \{1, 2, 3\}$ . (This is the Newtonian approximation, for  $GM/r \ll 1$ , of the metric tensor of a spherically symmetric body of mass  $M$ .) In your calculation neglect all terms quadratic in  $GM$ , if relevant.

In each case above: Try to find some (all?) solutions, and discuss the interesting features of the equations, if any.

2 [For self-study, will only be covered in class if time allows.]

i. Let  $X$  be a vector field which satisfies the following: for every affinely parameterised geodesic  $\gamma$  it holds that  $g(X, \dot{\gamma})$  is constant along  $\gamma$ . Show that this implies that  $X$  satisfies the following equation, known as *the Killing equation*:

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0 . \quad (2)$$

Solutions of the Killing equation are called *Killing vectors*.

ii. Show that a linear combination of Killing vectors with constant coefficients is a Killing vector. Thus, the set of Killing vectors forms a vector space.

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- iii. Let  $X$  be a Killing vector field, and let  $s \mapsto x^\mu(s)$  be an integral curve of  $X$ : by definition, this means that

$$\frac{dx^\mu}{ds}(s) = X^\mu(x(s)) . \quad (3)$$

In other words, the vector field  $X$  is tangent to the curves  $s \mapsto x^\mu(s)$ . Show that the integral curves of  $X$  are geodesics if and only we have

$$\nabla_\alpha(g(X, X)) = 0 \text{ along the curve } x(s). \quad (4)$$

- iv. It follows from Q2ii of PS11 that the hypothesis of point i. is satisfied by the vector fields  $\partial_t$  and  $\partial_\varphi$  for the Schwarzschild metric (1). Use this observation, and the remaining results above, to show that there exist geodesics in Schwarzschild space-time with tangent  $X = \partial_t + \Omega\partial_\varphi$ , for a constant  $\Omega$  which you will determine. [*Hint: Find the set where  $\nabla_\alpha(g(X, X)) = 0$ .*]