Reminder: The Christoffel symbols of the Levi-Civita connection are defined by the formula

$$\Gamma^a{}_{bc} \equiv \Gamma^a_{bc} := \tfrac{1}{2} g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}) \ .$$

A vector with components $(X^1, \ldots, X^n)$ will often be written as $X^1\partial_1 + \ldots + X^n\partial_n \equiv X^a\partial_a$. Thus, ∂_1 is the same as the vector with components $(1, 0, \ldots 0)$, etc.

Similarly, a covector $(\alpha_1, \ldots, \alpha_n)$ will be written as $\alpha_1 dx^1 + \ldots + \alpha_n dx^n \equiv \alpha_a dx^a$.

51 Christoffel symbols

Calculate the Christoffel symbols for a) the Euclidean metric on $\mathbb{R}^2$ in polar coordinates: $g = d\rho^2 + \rho^2 d\varphi^2$, and b) the unit round metric on S^2: $h = d\theta^2 + \sin^2\theta d\varphi^2$.

[Hint: Calculate first the Christoffel symbols for a metric of the form $dx^2 + e^{2f(x)}dy^2$.]

52 Euler-Lagrange equations for geodesics •0.1

•0.1: **ptc:** *this one is enough for a wh*

i. Show that the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = \frac{\partial L}{\partial x^a} \tag{1}$$

associated with the Lagrange function

$$L(x^c, \dot{x}^c) = \frac{1}{2} g_{ab}\dot{x}^a\dot{x}^b \tag{2}$$

can be written as

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0 \ . \tag{3}$$

ii. Show that if the metric does not explicitly depend upon a coordinate, say x^1, then $g(\dot{x}, \partial_1)$ is constant along every geodesic.

iii. Use (1) for the two-dimensional metrics g and h of Problem 52 to calculate again the associated Christoffel symbols.

iv. Calculate the Riemann tensor, the Ricci tensor and the Ricci scalar of the unit round metric on S^2, namely $g = d\theta^2 + \sin^2\theta d\varphi^2$.

v. Write down the geodesic equation for timelike geodesics in the “post-Newtonian metric”:

$$g_{00} = -\left(1 - \frac{2GM}{r}\right) , \quad g_{0i} = 0 \ , \quad g_{ij} = \left(1 + \frac{2GM}{r}\right)\delta_{ij} \ , \tag{4}$$

with $i, j \in \{1, 2, 3\}$. (This is the Newtonian approximation, for $GM/r \ll 1$, of the metric tensor of a spherically symmetric body of mass M.) In your calculations neglect all terms quadratic in GM.

53 Affinely parameterized geodesics

Solutions $x^a(t)$ of (3) are called *affinely parameterized geodesics*, and the parameter t is said to be *affine*.

i. For a vector field X defined along a curve $s \mapsto \gamma(s)$, set

$$\frac{DX^a}{Ds} := \frac{dX^a}{ds} + \Gamma^a{}_{bc}\dot{\gamma}^b X^c ,$$

where the $\Gamma^a{}_{bc}$'s are the Christoffel symbols of the Levi-Civita connection associated with the metric g. Show that

$$\frac{d(g(X,Y)\circ\gamma)}{ds} = g\left(\frac{DX}{Ds}, Y\right) + g\left(X, \frac{DY}{Ds}\right) .$$

ii. Let $x^a(\sigma)$ be a geodesic with an affine parameter σ and let $W^a = \mathrm{d}x^a/\mathrm{d}\sigma$. Show that:

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(g_{ab}W^a W^b\right) = 0.$$

Recall that a vector X is called *timelike* if $g(X,X) < 0$, *null* if $X \neq 0$ and $g(X,X) = 0$, and *spacelike* if $g(X,X) > 0$. Deduce that the type of the tangent vector does not change along a geodesic. Hence it is meaningful to consider timelike, spacelike, or null geodesics.

iii. A curve is called *timelike* if its tangent is timelike everywhere. A parameter τ along a timelike curve $\tau \mapsto x(\tau)$ is called *proper time* if $g(dx/d\tau, dx/d\tau) = -1$. Show that for a timelike geodesic as in point ii, σ is an affine function of proper time τ (this means that $\sigma = a\tau + b$, for some constants a, b).

54 Constants of motion for geodesics

Using the variational principle for geodesics, write down the geodesic equations, and give the obvious constants of motion (see Problem 52 (ii) and Problem 53 (ii)), for

a) the *Schwarzschild* metric

$$g_m = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{1}{1-\frac{2m}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{5}$$

(you might wish to do the calculation for a metric of the form

$$-e^{2f(r)}dt^2 + e^{-2f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) ,$$

and specialize the result to Schwarzschild at the end, as many other important metrics are of this form, so the result might be useful later). Show that a geodesic initially tangent to the equatorial plane ($\theta = \pi/2$) always remains in it.

b) The following *pp-wave* metric

$$g = dx^2 + dy^2 - 2du\,dv + H(u,x)du^2 \ .$$

c) The "post-Newtonian" metric (4). In your calculation neglect all terms quadratic in GM, if relevant.