Please follow the instructions on moodle concerning which problems should be used for the Kreuzerlliste.

We write $\det g_{\mu\nu}$ for the determinant of a matrix $(g_{\mu\nu})$, $g^{\mu\nu}$ for the matrix inverse to $g_{\alpha\beta}$ (thus, $g^{\mu\nu}g_{\nu\alpha} = \delta^\mu_\alpha$), and $\phi^A{}_{,\mu}$ for $\partial\phi^A/\partial x^\mu$.

47 Let $A_{\mu\nu}$ be an invertible matrix, with inverse $B^{\mu\nu}$. Show that

$$\frac{\partial(\det A_{\mu\nu})}{\partial A_{\rho\sigma}} = (\det A_{\mu\nu})B^{\rho\sigma} \,. \tag{1}$$

Deduce that for matrices with entries which are functions on space-time

$$\partial_\alpha \sqrt{\det A_{\mu\nu}} = \frac{1}{2}\sqrt{\det A_{\mu\nu}}B^{\rho\sigma}\partial_\alpha A_{\rho\sigma} \,. \tag{2}$$

48 Write-down the Euler-Lagrange equations, the canonical energy-momentum tensor

$$t^\mu{}_\nu = \frac{\partial\mathscr{L}}{\partial\phi^A{}_{,\mu}}\phi^A{}_{,\nu} - \mathscr{L}\delta^\mu_\nu \,,$$

and the symmetric energy-momentum tensor

$$T_{\mu\nu} = 2\frac{\partial\mathscr{L}}{\partial g^{\mu\nu}} \,,$$

for the following Lagrageans:

i. $\mathscr{L} = \frac{1}{2}\left(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + m^2\phi^2\right)\sqrt{-\det g_{\mu\nu}}$
(massive scalar field);

ii. $\mathscr{L} = \left(\frac{1}{8\pi}g^{\alpha\beta}g^{\mu\nu}F_{\alpha\mu}F_{\beta\nu}\right)\sqrt{-\det g_{\mu\nu}}$,
where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (Maxwell field); and

iii. $\mathscr{L} = \frac{1}{8\pi}\left(e^{-a\phi}g^{\alpha\beta}g^{\mu\nu}F_{\alpha\mu}F_{\beta\nu} + g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right)\sqrt{-\det g_{\mu\nu}}$,
where as before $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $a \in \mathbb{R}$ (dilaton-Maxwell system).

In the calculations of the Euler-Lagrange equations and of the canonical energy-momentum tensor you can assume that $g_{\mu\nu} = \eta_{\mu\nu}$, the Minkowski metric.

49 <u>Changes of coordinates</u>

Let $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$ be inertial coordinates on flat space-time, so the Minkowski metric has components

$$(g_{ab}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let X be the vector field which in the above coordinate system equals $(1, 1, 0, 0)$, and let α be a one-form which in the above coordinate system equals $(1, 1, 0, 0)$.

Find the metric coefficients $\tilde{g}_{ab}$, and the components of X and α, in each of the following coordinate systems.

(i) $\tilde{x}^0 = t - z,\ \ \tilde{x}^1 = r,\ \ \tilde{x}^2 = \theta,\ \ \tilde{x}^3 = z$
(ii) $\tilde{x}^0 = \tau,\ \ \tilde{x}^1 = \phi,\ \ \tilde{x}^2 = y,\ \ \tilde{x}^3 = z,$
(iii) $\tilde{x}^0 = t,\ \ \tilde{x}^1 = r,\ \ \tilde{x}^2 = \theta,\ \ \tilde{x}^3 = \phi$

where, in the first case, r, θ are plane polar coordinates in the x, y plane: $x = r\cos\theta$, $y = r\sin\theta$; in the second, τ, ϕ are 'Rindler coordinates', defined by $t = \tau\cosh\phi$, $x = \tau\sinh\phi$; and, in the third, r, θ, ϕ are spherical polar coordinates. In each case, state which region of Minkowski space the coordinate system covers. *[Hint: A quick method for the metric is to write it as* $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ *and substitute, for example,* $\mathrm{d}x = \cos\theta\mathrm{d}r - r\sin\theta\mathrm{d}\theta$*, and so on. Of course you should convince yourself that this is legitimate.]*

50 Lie bracket

Recall that vector fields can be identified with homogeneous linear first order partial differential operators $X = X^a\partial_a$ acting on functions as $X(f) = X^a\partial_a f$.

The Lie-bracket $[X, Y]$ of two vector fields X and Y is defined as

$$[X, Y](f) = X(Y(f)) - Y(X(f))\ .$$

Show that $[X, Y]$ also is a vector field, i.e. a homogeneous linear first order differential operator, with components

$$[X, Y]^a = X^b\partial_b Y^a - Y^b\partial_b X^a\ . \tag{3}$$

Check, by a direct coordinate calculation, that the right-hand-side of (3) transforms as a vector field under changes of coordinates.

[For self-study:] Prove the *Jacobi identity*:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0\ .$$