You are expected to cross three problems amongst exercises 30, 31, 32 and 33. As usual, you are strongly encouraged to attempt more than three.

30. The *alternating tensor* $\epsilon_{\alpha\beta\gamma\delta}$ is defined by the requirement that it changes sign under the permutation of any two indices (such tensors are called *totally antisymmetric*), and

$$\epsilon_{0123} = 1 \ .$$

Does this indeed define $\epsilon_{\alpha\beta\gamma\delta}$ uniquely? *[Hint: What is the value of $\epsilon_{\alpha\beta\gamma\delta}$ when some indices coincide?]*

Define $\epsilon^{\alpha\beta\gamma\delta}$ by raising the indices using *some* symmetric two-contravariant tensor $\eta^{\mu\nu}$, with inverse tensor $\eta_{\mu\nu}$, possibly, but not necessarily, equal to the Minkowski metric:

$$\epsilon^{\alpha\beta\gamma\delta} = \eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\gamma\rho}\eta^{\delta\sigma}\epsilon_{\mu\nu\rho\sigma} \ .$$

Show that $\epsilon^{\alpha\beta\gamma\delta}$ is totally antisymmetic. Explain why

$$\epsilon^{0123} = \det \eta^{\alpha\beta} \ .$$

[Hint: how would the corresponding equation look like in two-dimensions? and in three? If in doubt, look up the definition of the determinant on e.g. Wikipedia.] Similarly show that

$$\Lambda^{\alpha'}{}_{\alpha}\Lambda^{\beta'}{}_{\beta}\Lambda^{\gamma'}{}_{\gamma}\Lambda^{\delta'}{}_{\delta}\epsilon_{\alpha'\beta'\gamma'\delta'} = \det \Lambda \epsilon_{\alpha\beta\gamma\delta} \ .$$

How can this be generalised to other dimensions? or to the Euclidean metric? How many totally antisymmetric tensors with five indices are there in dimension n, $1 \leq n \leq 7$?

[A word of caution: we will see in the remainder of this course that ϵ is not quite a tensor, but a tensor density. It is, however, consistent to consider it as a tensor when considering only orthochronous orientation preserving Lorentz transformations.]

31. Let a bracket over indices denote complete antisymmetrisation, and a parenthesis over indices denote complete symmetrisation: for example,

$$A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu}) \ , \quad A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}) \ , \quad \delta^{[\alpha}_{\mu}\delta^{\gamma]}_{\rho} = \frac{1}{2}(\delta^{\alpha}_{\mu}\delta^{\gamma}_{\rho} - \delta^{\gamma}_{\mu}\delta^{\alpha}_{\rho}) \ ,$$

and similarly for 3, 4 or more indices (with combinatorial prefactors $1/n!$). Show that

i. $A_{(\mu\nu)} = A_{(\nu\mu)}$, $A_{[\mu\nu]} = -A_{[\nu\mu]}$,

ii. $A_{\mu\nu} = A_{[\mu\nu]} + A_{(\mu\nu)}$,

iii. $A^{[\mu\nu]}B_{\mu\nu} = A^{\mu\nu}B_{[\mu\nu]} = A^{[\mu\nu]}B_{[\mu\nu]}$,

iv. $A^{(\mu\nu)}B_{\mu\nu} = A^{\mu\nu}B_{(\mu\nu)} = A^{(\mu\nu)}B_{(\mu\nu)}$,

v. $A^{[\mu\nu\rho]}B_{\mu\nu\rho} = A^{\mu\nu\rho}B_{[\mu\nu\rho]}$,

vi. $\delta_\mu^{[\alpha}\delta_\rho^{\gamma]} = \delta_{[\mu}^\alpha\delta_{\rho]}^\gamma$,

vii. $\delta_\mu^{[\alpha}\delta_\nu^\beta\delta_\rho^{\gamma]} = \delta_{[\mu}^\alpha\delta_\nu^\beta\delta_{\rho]}^\gamma$,

viii. $\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\rho} = -6\delta_\rho^\delta$,

ix. $\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\nu\rho} = -4\delta_{[\nu}^\gamma\delta_{\rho]}^\delta$.

32. Assuming the tensorial transformation law of $F^{\mu\nu}$, derive the explicit formulae for the transformation laws of the electric and magnetic fields under a boost along the first coordinate axis.

33. Let $\vec{E}\cdot\vec{B} = 0$, and suppose that $|\vec{E}|^2 \neq |\vec{B}|^2$. Show that there exists a Lorentz frame in which either $\vec{E}$ or $\vec{B}$ vanishes. [*Hint: apply a boost with $\vec{v}$ proportional to $\vec{E}\times\vec{B}$.*]

34. Let $*F_{\alpha\beta} := \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}$. Show that the contraction $F_{\alpha\beta}F^{\alpha\beta}$ is invariant (more precisely, behaves as a scalar) under Lorentz transformations, while $F^{\alpha\beta}*F_{\alpha\beta}$ either remains invariant, or changes sign. Express those contractions in terms of $\vec{E}$ and $\vec{B}$.

35. [*For self-study, unlikely to be covered in class*]

a) Assuming that $\Lambda^\alpha{}_\beta$ is a Lorentz matrix, show that the matrix $A^\alpha{}_\nu := \eta^{\alpha\beta}\Lambda^\mu{}_\beta\eta_{\mu\nu}$ is inverse to $\Lambda^\alpha{}_\beta$.

b) Recall that we required that $F^{\mu\nu}$ transforms as a *two-contravariant tensor* under Lorentz transformations: if $\bar{x}^\alpha = \Lambda^\alpha{}_\beta x^\beta + a^\alpha$, then

$$\overline{F}^{\mu\nu}(\bar{x}) = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta F^{\alpha\beta}(x)\ ,$$

and that $F_{\mu\nu}$ has been defined as

$$F_{\mu\nu} := \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta}\ .$$

Use 1) to show that $F_{\alpha\beta}$ transforms as

$$\overline{F}_{\mu\nu}(\bar{x}) = (\Lambda^{-1})^\alpha{}_\mu(\Lambda^{-1})^\beta{}_\nu F_{\alpha\beta}(x)$$

(this is called the *transformation law of a two-covariant tensor*).