1 Some geodesics

Using the variational principle for geodesics, write down the geodesic equations, and give the obvious constants of motion (compare PS11 Q1ii and Q2ii), for

a) the *Schwarzschild* metric

$$g_m = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{1}{1 - \frac{2m}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

(you might wish to do the calculation for a metric of the form

$$-e^{2f(r)}dt^2 + e^{-2f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) ,$$

and specialize the result to Schwarzschild at the end, as many other important metrics are of this form, so the result might be useful later). Show that a geodesic initially tangent to the equatorial plane always remains in it.

b) The following *pp-wave* metric

$$g = dx^2 + dy^2 - 2du\,dv + H(u,x)du^2 .$$

c) The "post-Newtonian" metric

$$g_{00} = -(1 - \frac{2GM}{r}) , \quad g_{0i} = 0 , \quad g_{ij} = \left(1 + \frac{2GM}{r}\right)\delta_{ij} ,$$

with $i, j \in \{1,2,3\}$. (This is the Newtonian approximation, for $GM/r \ll 1$, of the metric tensor of a spherically symmetric body of mass M.) In your calculation neglect all terms quadratic in GM.

In each case above, try to find some (all?) solutions.

2 *[For self-study, will only be covered in class if time allows.]*

i. Let X be a vector field which satisfies the following: for every affinely parameterised geodesic γ it holds that $g(X,\dot\gamma)$ is constant along γ. Show that this implies that X satisfies the following equation, known as *the Killing equation*:

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0 . \tag{2}$$

Solutions of the Killing equation are called *Killing vectors*.

ii. Show that a linear combination of Killing vectors with constant coefficients is a Killing vector. Thus, the set of Killing vectors forms a vector space.

iii. Let X be a Killing vector field, and let $s \mapsto x^\mu(s)$ be an integral curve of X: by definition, this means that

$$\frac{dx^\mu}{ds}(s) = X^\mu(x(s)) . \tag{3}$$

In other words, the vector field X is tangent to the curves $s \mapsto x^\mu(s)$. Show that the integral curves of X are geodesics if and only we have

$$\nabla_\alpha(g(X,X)) = 0 \text{ along the curve } x(s). \tag{4}$$

iv. It follows from Q2ii of PS11 that the hypothesis of point i. is satisfied by the vector fields ∂_t and ∂_φ for the Schwarzschild metric (1). Use this observation, and the remaining results above, to show that there exist geodesics in Schwarzschild space-time with tangent $X = \partial_t + \Omega\partial_\varphi$, for a constant Ω which you will determine. *[Hint: Find the set where $\nabla_\alpha(g(X,X)) = 0$.]*