1 Christoffel symbols

The Christoffel symbols of the Levi-Civita connection are defined by the formula

$$\Gamma^a_{bc} = \tfrac{1}{2} g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}) \; .$$

Using this definition, calculate the Christoffel symbols for a) the Euclidean metric on $\mathbb{R}^2$ in polar coordinates: $g = d\rho^2 + \rho^2 d\varphi^2$, and b) the unit round metric on S^2: $h = d\theta^2 + \sin^2\theta d\varphi^2$.

i. Show that the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = \frac{\partial L}{\partial x^a} \tag{1}$$

associated with the Lagrange function

$$L(x^c, \dot{x}^c) = \frac{1}{2} g_{ab}\dot{x}^a \dot{x}^b \tag{2}$$

can be written as

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0 \; . \tag{3}$$

ii. Show that if the metric does not explicitly depend upon a coordinate, say x^1, then $g(\dot{x}, \partial_1)$ is constant along every geodesic.

iii. Use the explicit form of (1) for the two-dimensional metrics g and h above to calculate again the Christoffel symbols.

2 Geodesics

Solutions $x^a(t)$ of (3) are called *affinely parameterized geodesics*, and the parameter t is said to be *affine*.

i. For a vector field X defined along a curve $s \mapsto \gamma(s)$, set

$$\frac{DX^a}{ds} := \frac{dX^a}{ds} + \Gamma^a{}_{bc}\dot{\gamma}^b X^c \; ,$$

where the $\Gamma^a{}_{bc}$'s are the Christoffel symbols of the Levi-Civita connection associated with the metric g. Show that

$$\frac{d(g(X,Y)\circ\gamma)}{ds} = g(\frac{DX}{Ds}, Y) + g(X, \frac{DY}{Ds}) \; .$$

ii. Let $x^a(\sigma)$ be a geodesic with an affine parameter σ and let $W^a = \mathrm{d}x^a/\mathrm{d}\sigma$. Show that:

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}(g_{ab}W^a W^b) = 0.$$

Recall that a vector X is called *timelike* if $g(X,X) > 0$, *null* if $X \neq 0$ and $g(X,X) = 0$, and *spacelike* if $g(X,X) < 0$. Deduce that the type of the tangent vector does not change along a geodesic. Hence it is meaningful to consider timelike, spacelike, or null geodesics.

iii. A curve is called *timelike* if its tangent is timelike everywhere. A parameter τ along a timelike curve $\tau \mapsto x(\tau)$ is called *proper time* if $g(dx/d\tau, dx/d\tau) = -1$. Show that for a timelike geodesic as in point ii, σ is an affine function of proper time τ (this means that $\sigma = a\tau + b$, for some constants a, b).