1 Changes of coordinates

Let \( x^0 = t, \) \( x^1 = x, \) \( x^2 = y, \) \( x^3 = z \) be inertial coordinates on flat space-time, so the Minkowski metric has components

\[
(g_{ab}) = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Let \( X \) be the vector field which in the above coordinate system equals \((1, 1, 0, 0),\) and let \( \alpha \) be a one-form which in the above coordinate system equals \((1, 1, 0, 0).\) Find the metric coefficients \( \tilde{g}_{ab}, \) and the components of \( X \) and \( \alpha, \) in each of the following coordinate systems.

(i) \( \tilde{x}^0 = t - z, \) \( \tilde{x}^1 = r, \) \( \tilde{x}^2 = \theta, \) \( \tilde{x}^3 = z \)

(ii) \( \tilde{x}^0 = \tau, \) \( \tilde{x}^1 = \phi, \) \( \tilde{x}^2 = y, \) \( \tilde{x}^3 = z \)

(iii) \( \tilde{x}^0 = t, \) \( \tilde{x}^1 = r, \) \( \tilde{x}^2 = \theta, \) \( \tilde{x}^3 = \phi \)

where, in the first case, \( r, \theta \) are plane polar coordinates in the \( x, y \) plane: \( x = r \cos \theta, \) \( y = r \sin \theta; \) in the second, \( \tau, \phi \) are ‘Rindler coordinates’, defined by \( t = \tau \cosh \phi, \) \( x = \tau \sinh \phi; \) and, in the third, \( r, \theta, \phi \) are spherical polar coordinates. In each case, state which region of Minkowski space the coordinate system covers. [Hint: A quick method for the metric is to write it as \( ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \) and substitute, for example, \( dx = \cos \theta dr - r \sin \theta d\theta, \) and so on. Of course you should convince yourself that this is legitimate.]

2 Lie bracket

Recall that vector fields can be identified with homogeneous linear first order partial differential operators \( X = X^a \partial_a \) acting on functions as \( X(f) = X^a \partial_a f. \)

The Lie-bracket \([X, Y]\) of two vector fields \( X \) and \( Y \) is defined as

\[
[X, Y](f) = X(Y(f)) - Y(X(f)).
\]

Show that \([X, Y]\) also is a vector field, i.e. a homogeneous linear first order differential operator, with components

\[
[X, Y]^a = X^b \partial_b Y^a - Y^b \partial_b X^a.
\] (1)

Check, by a direct coordinate calculation, that the right-hand-side of (1) transforms as a vector field under changes of coordinates.

[For self-study:] Prove the Jacobi identity:

\[
[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.
\]