1 Changes of coordinates

Let $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$ be inertial coordinates on flat space-time, so the Minkowski metric has components

$$(g_{ab}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let X be the vector field which in the above coordinate system equals $(1, 1, 0, 0)$, and let α be a one-form which in the above coordinate system equals $(1, 1, 0, 0)$.

Find the metric coefficients $\tilde{g}_{ab}$, and the components of X and α, in each of the following coordinate systems.

(i) $\tilde{x}^0 = t - z$, $\tilde{x}^1 = r$, $\tilde{x}^2 = \theta$, $\tilde{x}^3 = z$
(ii) $\tilde{x}^0 = \tau$, $\tilde{x}^1 = \phi$, $\tilde{x}^2 = y$, $\tilde{x}^3 = z$,
(iii) $\tilde{x}^0 = t$, $\tilde{x}^1 = r$, $\tilde{x}^2 = \theta$, $\tilde{x}^3 = \phi$

where, in the first case, r, θ are plane polar coordinates in the x, y plane: $x = r\cos\theta$, $y = r\sin\theta$; in the second, τ, ϕ are 'Rindler coordinates', defined by $t = \tau\cosh\phi$, $x = \tau\sinh\phi$; and, in the third, r, θ, ϕ are spherical polar coordinates. In each case, state which region of Minkowski space the coordinate system covers. *[Hint: A quick method for the metric is to write it as* $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ *and substitute, for example,* $\mathrm{d}x = \cos\theta\mathrm{d}r - r\sin\theta\mathrm{d}\theta$*, and so on. Of course you should convince yourself that this is legitimate.]*

2 Lie bracket

Recall that vector fields can be identified with homogeneous linear first order partial differential operators $X = X^a\partial_a$ acting on functions as $X(f) = X^a\partial_a f$.

The Lie-bracket $[X, Y]$ of two vector fields X and Y is defined as

$$[X, Y](f) = X(Y(f)) - Y(X(f)) .$$

Show that $[X, Y]$ also is a vector field, i.e. a homogeneous linear first order differential operator, with components

$$[X, Y]^a = X^b\partial_b Y^a - Y^b\partial_b X^a . \quad (1)$$

Check, by a direct coordinate calculation, that the right-hand-side of (1) transforms as a vector field under changes of coordinates.

[For self-study:] Prove the *Jacobi identity*:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 .$$