1 (“Raising and lowering of indices”) Define
\[ B_\alpha := \eta_{\alpha\beta} A_\beta, \quad C^\gamma := \eta^{\gamma\sigma} B_\sigma. \] (1)

Show that
\[ C^\gamma = A^\gamma. \] (2)

The first operation in (1) is called “lowering an index with the metric”; the second “raising an index with the metric”. What does (2) say about this operation?

From now on we shall simply write
\[ A_\alpha := \eta_{\alpha\beta} A_\beta, \quad B^\gamma := \eta^{\gamma\sigma} B_\sigma. \]

Show that
\[ A_\alpha B^\alpha = A^\alpha B_\alpha. \]

2 Define the \(*\)-operation on anti-symmetric tensors as
\[ *F_{\alpha\beta} := \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} F_{\gamma\delta}, \quad *F^{\alpha\beta} := \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}, \]
where
\[ \epsilon_{\alpha\beta\gamma\delta} := \eta_{\alpha\mu} \eta_{\beta\nu} \epsilon^{\mu\nu\gamma\delta}. \]

How does this differ from the definition given in the lecture?

i. Show that
\[ *F_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta}. \]

ii. Show that the double star of an anti-symmetric tensor is the negative of this tensor.

iii. Show that if \( F_{\mu\nu} \) is anti-symmetric, then \( *F^{\alpha\beta} *F_{\alpha\beta} = -F^{\alpha\beta} F_{\alpha\beta} \).

3 Consider a charged particle which moves along a straight line in Minkowski space-time in an electromagnetic field. Show that its velocity \( \vec{v} \) satisfies \( \vec{E} + \vec{v} \times \vec{B} = 0 \) and \( \vec{E} \cdot \vec{v} = 0 \). Find all possible solutions for \( \vec{v} \) in terms of \( \vec{E} \) and \( \vec{B} \). What can you say about \( F^{\alpha\beta} F_{\alpha\beta} \) and \( *F^{\alpha\beta} F_{\alpha\beta} \)?

4 [For self study:] Let \( T_{\mu\nu} \) be the energy-momentum tensor of the electromagnetic field. Express \( T_{00}, T_{0i}, \) and \( T_{ij} \) in terms of \( E^i \) and \( B^j \).

Show that
\[ T_{\mu\nu} T^\nu = \frac{1}{4} T_{\alpha\beta} T^{\alpha\beta} \eta_{\mu\nu}. \]

5 [For self study:] Describe the gauge transformations which preserve the Lorenz gauge condition.