1 **(“Raising and lowering of indices”)** Define

$$B_\alpha := \eta_{\alpha\beta} A^\beta \;, \quad C^\gamma := \eta^{\gamma\sigma} B_\sigma \;. \tag{1}$$

Show that

$$C^\gamma = A^\gamma \;. \tag{2}$$

The first operation in (1) is called “lowering an index with the metric”; the second “raising an index with the metric”. What does (2) say about this operation?

From now on we shall simply write

$$A_\alpha := \eta_{\alpha\beta} A^\beta \;, \quad B^\gamma := \eta^{\gamma\sigma} B_\sigma \;.$$

Show that

$$A_\alpha B^\alpha = A^\alpha B_\alpha \;.$$

2 Define the $*$-operation on anti-symmetric tensors as

$$*F_{\alpha\beta} := \frac{1}{2} \epsilon_{\alpha\beta}{}^{\gamma\delta} F_{\gamma\delta} \;, \qquad *F^{\alpha\beta} := \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \;,$$

where

$$\epsilon_{\alpha\beta}{}^{\gamma\delta} := \eta_{\alpha\mu}\eta_{\beta\nu}\epsilon^{\mu\nu\gamma\delta} \;.$$

How does this differ from the definition given in the lecture?

i. Show that

$$*F_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta} \;.$$

ii. Show that if $F_{\mu\nu}$ is anti-symmetric, then $*F^{\alpha\beta} * F_{\alpha\beta} = -F^{\alpha\beta} F_{\alpha\beta}$.

iii. Show that the double star of an anti-symmetric tensor is the negative of this tensor.

3 Consider a charged particle which moves *along a straight line* in an electromagnetic field. Show that its velocity $\vec{v}$ satisfies $\vec{E} + \vec{v} \times \vec{B} = 0$ and $\vec{E} \cdot \vec{v} = 0$. Find all possible solutions for $\vec{v}$ in terms of $\vec{E}$ and $\vec{B}$. What can you say about $F^{\alpha\beta} F_{\alpha\beta}$ and $*F^{\alpha\beta} F_{\alpha\beta}$?

4 Let $T_{\mu\nu}$ be the energy-momentum tensor of the electromagnetic field. Express T_{00}, T_{0i}, and T_{ij} in terms of E^i and B^j.

[For self study: Show that

$$T_{\mu\rho} T^\rho{}_\nu = \frac{1}{3} T_{\alpha\beta} T^{\alpha\beta} \eta_{\mu\nu} \;.]$$

5 Describe the gauge transformations which preserve the Lorenz gauge condition.