1. Let a bracket over indices denote complete antisymmetrisation, and a parenthesis over indices denote complete symmetrisation: for example,

$$A_{[\mu \nu]} = \frac{1}{2} (A_{\mu \nu} - A_{\nu \mu}) , \quad A_{(\mu \nu)} = \frac{1}{2} (A_{\mu \nu} + A_{\nu \mu}) , \quad \delta_{\mu \rho}^{[\alpha \beta]} = \frac{1}{2} (\delta_{\mu \rho}^{\alpha \beta} - \delta_{\mu \rho}^{\beta \alpha}) ,$$

and similarly for 3, 4 or more indeces. Show that

i. $$A_{[\mu \nu]} B_{\mu \nu} = A_{\mu \nu} B_{(\mu \nu)} ,$$

ii. $$A_{[\mu \nu \rho]} B_{\mu \nu \rho} = A_{\mu \nu \rho} B_{(\mu \nu \rho)} ,$$

iii. $$\delta_{\mu \rho}^{[\alpha \beta]} = \delta_{\mu \rho}^{\alpha \beta} ,$$

iv. $$\delta_{\mu \rho}^{[\alpha \beta \gamma]} = \delta_{\mu \rho}^{\alpha \beta \gamma} ,$$

v. $$\epsilon^{\alpha \beta \gamma \delta} \epsilon_{\alpha \beta \gamma \delta} = 6 \delta_{\mu \rho} ,$$

vi. $$\epsilon^{\alpha \beta \gamma \delta} \epsilon_{\alpha \beta \gamma \delta} = -4 \delta_{\mu \rho} ,$$

2. Assuming the tensorial transformation law of $$F^{\mu \nu}$$, derive the explicit formulae for the transformation laws of the electric and magnetic fields under a boost along the first coordinate axis.

3. 1) Assuming that $$\Lambda^\alpha_\beta$$ is a Lorentz matrix, show that $$\eta^{\alpha \beta} \Lambda^\mu_\beta \eta_{\mu \nu}$$ is inverse to $$\Lambda^\alpha_\beta$$.  

2) Recall that we required that $$F^{\mu \nu}$$ transforms as follows under Lorentz transformations: if $$\bar{x}^\alpha = \Lambda^\alpha_\beta x^\beta + a^\alpha$$, then

$$F^{\mu \nu}(\bar{x}) = \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha \beta}(x) ,$$

and that $$F_{\mu \nu}$$ has been defined as

$$F_{\mu \nu} := \eta_{\mu \alpha} \eta_{\nu \beta} F^{\alpha \beta} .$$

Use 1) to show that $$F_{\alpha \beta}$$ transforms as

$$\bar{F}_{\mu \nu}(\bar{x}) = (\Lambda^{-1})^\mu_\nu (\Lambda^{-1})^\alpha_\rho F_{\alpha \beta}(x)$$

(this is called the transformation law of a two-covariant tensor);

4. Show that contractions $$F_{\alpha \beta} F^{\alpha \beta}$$ and $$F^{\alpha \beta} * F_{\alpha \beta}$$ are invariant (more precisely, behave as scalars) under Lorentz transformations. Express them in terms of $$\vec{E}$$ and $$\vec{B}$$.

5. Let $$\vec{E} \cdot \vec{B} = 0$$, and suppose that $$|\vec{E}|^2 \neq |\vec{B}|^2$$. Show that there exists a Lorentz frame in which either $$\vec{E}$$ or $$\vec{B}$$ vanishes.