1 Some geodesics

Using the variational principle for geodesics, write down the geodesic equations, and give the obvious constants of motion, for

a) the *Schwarzschild* metric

\[ g_m = - \left( 1 - \frac{2m}{r} \right) dt^2 + \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

(you might wish to do the calculation for a metric of the form

\[ -e^{2f(r)} dt^2 + e^{-2f(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) , \]

and specialize the result to Schwarzschild at the end, as many other important metrics are of this form, so the result might be useful later). Show that a geodesic initially tangent to the equatorial plane always remains in it.

b) The following *pp-wave* metric

\[ g = dx^2 + dy^2 - 2du dv + H(u, x) du^2 . \]

c) The “post-Newtonian” metric

\[ g_{00} = -(1 - \frac{2GM}{r}) , \quad g_{0i} = 0 , \quad g_{ij} = \left( 1 + \frac{2GM}{r} \right) \delta_{ij} , \]

with \( i, j \in \{1, 2, 3\} \). (This is the Newtonian approximation, for \( GM/r \ll 1 \), of the metric tensor of a spherically symmetric body of mass \( M \).) In your calculation neglect all terms quadratic in \( GM \).