

- 1 (“Raising and lowering of indices”) Define

$$B_\alpha := \eta_{\alpha\beta} A^\beta, \quad C^\gamma := \eta^{\gamma\sigma} B_\sigma. \quad (1)$$

Show that

$$C^\gamma = A^\gamma. \quad (2)$$

The first operation in (1) is called “lowering an index with the metric”; the second “raising an index with the metric”. What does (2) say about this operation?

From now on we shall simply write

$$A_\alpha := \eta_{\alpha\beta} A^\beta, \quad B^\gamma := \eta^{\gamma\sigma} B_\sigma.$$

Show that

$$A_\alpha B^\alpha = A^\alpha B_\alpha.$$

- 2 Define the \*-operation on anti-symmetric tensors as

$$*F_{\alpha\beta} := \frac{1}{2} \epsilon_{\alpha\beta}{}^{\gamma\delta} F_{\gamma\delta}, \quad *F^{\alpha\beta} := \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta},$$

where

$$\epsilon_{\alpha\beta}{}^{\gamma\delta} := \eta_{\alpha\mu} \eta_{\beta\nu} \epsilon^{\mu\nu\gamma\delta}.$$

How does this differ from the definition given in the lecture?

- i. Show that

$$*F_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta}.$$

- ii. Show that if  $F_{\mu\nu}$  is anti-symmetric, then  $*F^{\alpha\beta} *F_{\alpha\beta} = -F^{\alpha\beta} F_{\alpha\beta}$ .  
 iii. Show that the double star of an anti-symmetric tensor is the negative of this tensor.

- 3 Describe all gauge transformations which preserve the Lorenz gauge condition, if any.
- 4 Consider a charged particle which moves *along a straight line* in an electromagnetic field. Show that its velocity  $\vec{v}$  satisfies  $\vec{E} + \vec{v} \times \vec{B} = 0$  and  $\vec{E} \cdot \vec{v} = 0$ . Find all possible solutions for  $\vec{v}$  in terms of  $\vec{E}$  and  $\vec{B}$ . What can you say about  $F^{\alpha\beta} F_{\alpha\beta}$  and  $*F^{\alpha\beta} F_{\alpha\beta}$ ?
- 5 Let  $T_{\mu\nu}$  be the energy-momentum tensor of the electromagnetic field. Express  $T_{00}$ ,  $T_{0i}$ , and  $T_{ij}$  in terms of  $E^i$  and  $B^j$ . Show that

$$T_{\mu\rho} T^\rho{}_\nu = \frac{1}{3} T_{\alpha\beta} T^{\alpha\beta} \eta_{\mu\nu}.$$