1 Let a bracket over indices denote complete antisymmetrisation, and a parenthesis over indices denote complete symmetrisation: for example,

$$A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\mu\nu}) \ , \quad A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\mu\nu}) \ , \quad \delta^{[\alpha}_{\mu}\delta^{\gamma]}_{\rho} = \frac{1}{2}(\delta^{\alpha}_{\mu}\delta^{\gamma}_{\rho} - \delta^{\gamma}_{\mu}\delta^{\alpha}_{\rho}) \ ,$$

etc. Show that

i. $A^{[\mu\nu]}B_{\mu\nu} = A^{\mu\nu}B_{[\mu\nu]}$, $A^{(\mu\nu)}B_{\mu\nu} = A^{\mu\nu}B_{(\mu\nu)}$,

ii. $A^{[\mu\nu\rho]}B_{\mu\nu\rho} = A^{\mu\nu\rho}B_{[\mu\nu\rho]}$,

iii. $\delta^{[\alpha}_{\mu}\delta^{\gamma]}_{\rho} = \delta^{\alpha}_{[\mu}\delta^{\gamma}_{\rho]}$,

iv. $\delta^{[\alpha}_{\mu}\delta^{\beta}_{\nu}\delta^{\gamma]}_{\rho} = \delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu}\delta^{\gamma}_{\rho]}$,

v. $\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\rho} = -6\delta^{\delta}_{\rho}$,

vi. $\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\nu\rho} = -4\delta^{\gamma}_{[\nu}\delta^{\delta}_{\rho]}$.

2 Assuming the tensorial transformation law of $F^{\mu\nu}$, derive the explicit formulae for the transformation laws of the electric and magnetic fields under a boost along the first coordinate axis.

3 1) Assuming that $\Lambda^{\alpha}{}_{\beta}$ is a Lorentz matrix, show that $\eta^{\alpha\beta}\Lambda^{\mu}{}_{\beta}\eta_{\mu\nu}$ is inverse to $\Lambda^{\alpha}{}_{\beta}$.

2) Recall that we required that $F^{\mu\nu}$ transforms as follows under Lorentz transformations: if $\bar{x}^{\alpha} = \Lambda^{\alpha}{}_{\beta}x^{\beta} + a^{\alpha}$, then

$$\overline{F}^{\mu\nu}(\bar{x}) = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta}(x) \ ,$$

and that $F_{\mu\nu}$ has been defined as

$$F_{\mu\nu} := \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta} \ .$$

Use 1) to show that

a) $F_{\alpha\beta}$ transforms as

$$\overline{F}_{\mu\nu}(\bar{x}) = (\Lambda^{-1})^{\alpha}{}_{\mu}(\Lambda^{-1})^{\beta}{}_{\nu}F_{\alpha\beta}(x)$$

(this is called the *transformation law of a two-covariant tensor*);

b) $F_{\alpha\beta}F^{\alpha\beta}$ and $F^{\alpha\beta} *F_{\alpha\beta}$ are invariant (more precisely, behave as scalars) under Lorentz transformations.

4 Express $F_{\alpha\beta}F^{\alpha\beta}$ and $F^{\alpha\beta} *F_{\alpha\beta}$ in terms of $\vec{E}$ and $\vec{B}$.

5 Let $\vec{E} \cdot \vec{B} = 0$, and suppose that $|\vec{E}|^2 \neq |\vec{B}|^2$. Show that there exists a a Lorentz frame in which either $\vec{E}$ or $\vec{B}$ vanishes.